Perturbative Gravity and Gauge-Theory Relations

Søndergaard, Thomas

Publication date:
2012

Document version
Early version, also known as pre-print

Citation for published version (APA):
Søndergaard, T. (2012). *Perturbative Gravity and Gauge-Theory Relations*. (Københavns Universitet, Niels Bohr Institute ed.).

FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

PhD Thesis

Thomas Søndergaard

Perturbative Gravity and Gauge-Theory Relations

Academic advisors:

Poul Henrik Damgaard
Niels Emil Jannik Bjerrum-Bohr

Submitted: August 20, 2012

Perturbative Gravity and Gauge-Theory Relations

by

Thomas Søndergaard

Submitted for the degree of Doctor of Philosophy in Physics
at the Niels Bohr Institute, Faculty of Science,
University of Copenhagen.

August 2012

Abstract

This thesis is concerned with some of the surprising relations that exist for scattering amplitudes in (supersymmetric) gauge and gravity theories. Recently a new class of gauge-theory relations, called the Bern-Carrasco-Johansson (BCJ) relations, were discovered. They reduce the number of independent color-ordered tree-level amplitudes down to $(n-3)!$. In this thesis these relations are considered both in the context of string and field theory. In string theory they follow from monodromy relations between open-string amplitudes, while in field theory they can either be derived from recursion relations or seen as a consequence of a color-kinematic duality. We then investigate the Kawai-Lewellen-Tye (KLT) relations, which express closed-string amplitudes as products between open-string amplitudes. From a traditional Feynman diagrammatic point of view their validity in the field-theory limit is very unexpected. However, we show how these relations can be derived in field theory from the analytical properties of general n-point amplitudes. An important tool in this derivation will be the Britto-Cachazo-Feng-Witten recursion relations. Further examination will show that the KLT-relations also contain pure gauge-theory relations (vanishing identities) when the product is taken between amplitudes belonging to different helicity sectors. The KLT-relations have a natural extension to maximally supersymmetric gravity and Yang-Mills theories. It will be shown how to reduce to less supersymmetric theories, and map out the resulting gravity theories. In this formalism the vanishing identities follow naturally due to violation of either $SU(\mathcal{N})$ invariance or breaking of $U(1)$-charge conservation. In the last chapter we will explore some of the progress that has been made in extending these structures to loop level.

List of Publications

1. T. Sondergaard, **New Relations for Gauge-Theory Amplitudes with Matter**, Nucl. Phys. B **821** (2009) 417 [arXiv:0903.5453 [hep-th]].

2. N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, **Monodromy and Jacobi-like Relations for Color-Ordered Amplitudes**, JHEP **1006** (2010) 003 [arXiv:1003.2403 [hep-th]].

3. N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, **Gravity and Yang-Mills Amplitude Relations**, Phys. Rev. D **82** (2010) 107702 [arXiv:1005.4367 [hep-th]].

4. N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, **New Identities among Gauge Theory Amplitudes**, Phys. Lett. B **691** (2010) 268 [arXiv: 1006.3214 [hep-th]].

5. N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, **Proof of Gravity and Yang-Mills Amplitude Relations**, JHEP **1009** (2010) 067 [arXiv:1007.3111 [hep-th]].

6. N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, **The Momentum Kernel of Gauge and Gravity Theories**, JHEP **1101** (2011) 001 [arXiv:1010.3933 [hep-th]].

7. N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, **Unusual identities for QCD at tree-level**, J. Phys. Conf. Ser. **287** (2011) 012030 [arXiv: 1101.5555 [hep-ph]].

8. N. E. J. Bjerrum-Bohr, P. H. Damgaard, H. Johansson and T. Sondergaard, **Monodromy–like Relations for Finite Loop Amplitudes**, JHEP **1105** (2011) 039 [arXiv: 1103.6190 [hep-th]].

9. T. Sondergaard, **Perturbative Gravity and Gauge Theory Relations: A Review**, Adv. High Energy Phys. **2012** (2012) 726030 [arXiv:1106.0033 [hep-th]].

10. P. H. Damgaard, R. Huang, T. Sondergaard and Y. Zhang, **The Complete KLT-Map Between Gravity and Gauge Theories**, accepted for publication in JHEP (2012) [arXiv:1206.1577 [hep-th]].

Contents

Chapter 1

Introduction

The research field of scattering amplitudes is currently in the middle of a revolution. Developments within the last decade have shown that amplitudes are much simpler objects than expected from the traditional Feynman diagrammatic description. The usual text-book approach for the calculation of an amplitude is to draw all possible space-time processes, translate them to mathematical quantities by means of Feynman rules and add the whole thing up. Although this, in principle, enables one to calculate any amplitude, in practice it can be extremely complicated. For example, for a tree-level scattering amplitude involving 4 gluons there would be 4 diagrams, for 5 gluons 25 diagrams, for 6 gluons 220 diagrams, for 7 gluons 2485 diagrams *etc.* [1, 2]. Despite this rapid growth in the number of terms, already early on striking examples of the simplifications that can occur when adding up all contributing diagrams were known. Most famous of these is the Parke-Taylor Maximally-Helicity-Violating (MHV) formula from 1986 [3] (we will see explicit examples soon). However, for many the revolution really started in late 2003. In this year Witten showed how to describe perturbative gauge theory as a string theory in twistor space [4], highlighting the simplicity in the structure of scattering amplitudes. This was a hint at the existence of smarter methods to perform amplitude calculations.

Soon after Witten's paper such a method was formalized in the Cachazo-Svrcek-Witten (CSW) formalism [5, 6], in which tree amplitudes are calculated from vertices made up of an off-shell continuation of MHV amplitudes. Shortly after, in 2005, followed the Britto-Cachazo-Feng-Witten (BCFW) recursion relation [7, 8], calculating general n-point tree amplitudes in terms of lower-point *on-shell* amplitudes. This recursion relation is also valid for gravity theories [9] and can be naturally extended to supersymmetric theories as well [10, 11].

In addition, new relations and structures have been found in recent years. Most relevant to this thesis is a class of color-ordered subamplitude relations that were discovered in 2008, when Bern, Carrasco and Johansson (BCJ) found a curious *color-kinematic duality* for gauge-theory amplitudes [12]. By means of this duality they were able to write down new relations, reducing the number of independent n-point color-ordered gauge-theory amplitudes from $(n-2)!$, as dictated by the Kleiss-Kuijf (KK) relations [1, 13], down to $(n-3)!$. At this point these *BCJ-relations*, and their supersymmetric extension [14], had not been rigorously proven for general n points. The first proof came one year later from string theory, where Bjerrum-Bohr, Damgaard and Vanhove [15] and Stieberger [16] used monodromy to derive the $(n-3)!$ basis for color-ordered open-string amplitudes. In the field-theory limit these *monodromy relations* reduce exactly to KK- and BCJ-relations. In 2010 the BCJ-relations were then also proven from pure field theory [17, 18], using the BCFW recursion relations.

These new insights about the structure of scattering amplitudes has also helped to better understand earlier discoveries. By factorizing a closed string into a sum of products between two open strings, Kawai, Lewellen and Tye (KLT) derived in 1985 amazing relations between gravity and gauge-theory tree-level amplitudes [19]. These relations are satisfied to all orders in α', even in the field-theory limit $\alpha' \to 0$ [20, 21]. That the KLT-relations remain valid in this limit has been a puzzle for many years. As will be illustrated later, from a Lagrangian point of view, any connection between perturbative Einstein gravity and Yang-Mills theory seems very mysterious. Not until 2010, *i.e.* 25 years after their discovery, these relations could finally be proven from pure field theory [22, 23]. A spin-off of this investigation was the realization that the KLT-product between gauge-theory amplitudes can also be used to obtain purely gauge-theoretic relations ("vanishing identities") when the gauge-theory amplitudes belong to different helicity sectors [24, 25]. Soon after, this was explained using maximally supersymmetric theories. From this point of view the vanishing identities follow from KLT-products corresponding to $SU(8)_R$ violating gravity amplitudes [26, 27]. In 2012, similar arguments for less than maximally supersymmetric theories were given. In these cases the breaking of $U(1)$-charges are involved too [28]. The field theory proof of the KLT-relations also revealed the close connection between these and the above mentioned BCJ-relations.

We have here given a brief historical outline of the subjects that are investigated in this thesis. In particular we have tried to keep track of the correct chronological order in which developments occurred. This will not be continued throughout the thesis. Instead we take the liberty to present the material in the way we find most clear and rewarding. Note that this introduction does not do justice to the amount of incredible work done and the people involved in this field of research. We hope to at least partly make up for this as we go into the details below.

The thesis is structured as follows; in chapter 2 we introduce theoretical tools, notations and conventions used throughout this thesis. We also review several well-known aspects of scattering amplitudes in gauge and gravity theories that will set the stage for later chapters.

In chapter 3 we show how to derive a new class of gauge-theory amplitude relations. This will be done both from the point of view of string theory (monodromy relations) and from field theory (BCJ-relations). We also look into a special kind of parametrization of amplitudes which is intimately linked to the discovery of these new relations.

In chapter 4 we factorize closed-string amplitudes into products of open-string amplitudes glued together by a momentum kernel. We will see how these KLT-relations can be written in several different ways depending on how exactly the factorization is carried out.

In chapter 5 we will go to the field-theory limit of the KLT-relations, relating graviton amplitudes to the product of gluon amplitudes. The main goal of this chapter will be to give a purely field theoretic proof of the KLT-relations. This requires the use of the BCJ-relations considered in chapter 3, the above mentioned "vanishing identities", and an alternative way of writing gravity amplitudes in terms of an off-shell expression. At the end of this chapter we will also express the KLT-relations in terms of the parametrization from chapter 3.

In chapter 6 we extend the KLT-relations to supersymmetric theories. We will focus on the mapping between products of minimal super-Yang-Mills theories of varying degree of supersymmetry and supergravity theories. We will also see how the vanishing identities introduced in chapter 5 follow naturally from a maximally supersymmetric setting, and how to explain these identities from a less than maximally supersymmetric point of view.

In chapter 7 we review some of the extensions to loop level of the amplitude relations considered in this thesis. This chapter is meant to give a little taste of the research going on in this direction.

Finally in chapter 8 we summarise the results of this thesis and make some final comments.

Chapter 2

Preliminaries

This chapter has three main objectives. First, it introduces concepts and notation for later discussions. Second, it serves as a reference point that will hopefully lead to a greater appreciation of the work presented in this thesis. And Third, it will motivate the existence of some of the simplifying structures we will uncover.

Let us start off gently by reviewing some of the very basics about Yang-Mills theory.

2.1 Yang-Mills Theory

The Yang-Mills theory with a $SU(N)$ gauge group is described by the Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})\,, \tag{2.1}$$

with the field strength $F_{\mu\nu}$ given in terms of traceless hermitian $N \times N$ matrices of (gauge) fields A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\,. \tag{2.2}$$

We can write both $A_\mu = A_\mu^a T^a$ and $F_{\mu\nu} = F_{\mu\nu}^a T^a$ with T^a, $a = 1, \ldots, N^2 - 1$, being the generators of the $SU(N)$ group. These generators can be normalized as

$$\mathrm{Tr}[T^a T^b] = \delta^{ab}, \qquad [T^a, T^b] = i\sqrt{2} f^{abc} T^c\,, \tag{2.3}$$

where f^{abc} are the structure constants and δ^{ab} the Kronecker delta. Also note that

$$\widetilde{f}^{abc} \equiv \sqrt{2} f^{abc} = -i\mathrm{Tr}\big(T^a[T^b, T^c]\big)\,. \tag{2.4}$$

The A_μ^a are the actual gauge fields (representing gauge bosons) which transform in the adjoint representation of the $SU(N)$ group. We see that

$$\begin{aligned} F_{\mu\nu}^c T^c &= (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c)T^c - igA_\mu^a A_\nu^b [T^a, T^b] \\ &= (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g\sqrt{2} f^{abc} A_\mu^a A_\nu^b)T^c\,, \end{aligned} \tag{2.5}$$

hence $F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g\sqrt{2} f^{abc} A_\mu^a A_\nu^b$ and

$$\mathcal{L}_{YM} = -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a\,, \tag{2.6}$$

which can be written out in terms of the gauge fields A^a_μ

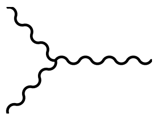
$$\mathcal{L}_{YM} = \frac{1}{2}A^{a\mu}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} - g\sqrt{2}f^{abc}A^{\mu a}A^{\nu b}\partial_\mu A^c_\nu - g^2\frac{1}{2}f^{abc}f^{a'b'c}A^{a\mu}A^{b\nu}A^{a'}_\mu A^{b'}_\nu \,. \tag{2.7}$$

Here $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ is just the usual Minkowski metric. From this Lagrangian we can obtain the Feynman rules

μ ν

$$= -\frac{i}{p^2}\eta_{\mu\nu}\,, \tag{2.8}$$

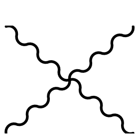
a, μ, k

 c, ρ, q

b, ν, p

$$= g\sqrt{2}f^{abc}\left[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\right]\,, \tag{2.9}$$

a, μ b, ν



d, σ c, ρ

$$\begin{aligned} = -i2g^2\Big[&f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ &+ f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\Big]\,, \end{aligned} \tag{2.10}$$

which are here presented in the Lorentz-Feynman gauge. The wavy lines represent gluons and we have omitted ghosts since they will be of no concern to us.

In principle we could now begin calculating scattering amplitudes by drawing all Feynman diagrams for a specific scattering process and using the above rules to translate them into the corresponding mathematical expressions. We would of course also need polarization vectors for the external legs, which have not been introduced yet. However, such amplitude calculations rapidly lead to unmanageable large expressions. One of the reasons, which should be obvious from the above Feynman rules, is the proliferation of color and space-time indices. This will obscure intermediate steps, making even tree-level calculations an extremely cumbersome task. In order to regain some control it is therefore custom to introduce the concept of color-ordering. Since this is by now almost text-book material the following discussion will be rather brief, see *e.g.* [29, 30].

Color Decompositions

The strategy is to decompose the full tree amplitude of the $SU(N)$ gauge theory into (simpler) subamplitudes completely independent of color indices.

As already mentioned the gluons live in the adjoint representation, *i.e.* they carry a color index $a = 1, 2, \ldots, N^2-1$. From the Feynman rules we see that a generic tree-level diagram contains a factor of f^{abc} for each three-vertex and a contracted pair $f^{abc}f^{cde}$ for each four-vertex. However, many of these indices will be contracted through the propagators such that only the n color indices associated with external states are left uncontracted. Using eq. (2.4) we can write the color part of a generic Feynman diagram in terms of products of $\mathrm{Tr}[T^a T^b T^c]$ factors with contractions over the internal color indices. With the identity

$$(T^a)_{i_1}^{\;\bar{\jmath}_1}(T^a)_{i_2}^{\;\bar{\jmath}_2} = \delta_{i_1}^{\;\bar{\jmath}_2}\delta_{i_2}^{\;\bar{\jmath}_1} - \frac{1}{N_c}\delta_{i_1}^{\;\bar{\jmath}_1}\delta_{i_2}^{\;\bar{\jmath}_2}\,, \tag{2.11}$$

these contracted T^a's can then be turned into *single-trace* terms. This allows us to decompose any n-point gluonic tree-level scattering amplitude into the form

$$\mathcal{A}_n = g^{n-2} \sum_{\sigma(2,3,\ldots,n)} \mathrm{Tr}[T^{a_1}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n)}}]A_n(1,\sigma(2),\ldots,\sigma(n))\,. \tag{2.12}$$

Here g is the gauge coupling, $i = 1,\ldots,n$ is shorthand for the momenta p_i, and A_n are what is called color-ordered subamplitudes. Basically they are just the kinematic coefficients of the corresponding single-trace terms. The sum runs over all permutations of leg $2,\ldots,n$, since we can use the cyclicity of traces to fix leg 1. For simplicity we have suppressed the helicity label on each of the external states.

Representing the full tree-level amplitude in this way the color and kinematic parts are completely separated. And because of the division into color factors depending on the ordering of legs (since traces of a string of matrices depend, up to cyclicity, on the order of the matrices), the corresponding kinematic coefficients, *i.e.* the subamplitudes, will be labeled by the same ordering. Following how the kinematic terms distribute when collecting into terms with common trace factor, color-ordered Feynman rules can be derived, from which the A_n's can be directly calculated. These can, for instance, be found in [29]. In this way only *planar* diagrams have to be considered. Non-planar diagrams are just planar diagrams with a different ordering, and hence, under the above decomposition, kinematic contributions from non-planar diagrams end up in the subamplitudes where their ordering correspond to planar diagrams.

There exists an alternative decomposition, written in terms of the structure constants $\widetilde{f}^{abc}$ instead, namely

$$\mathcal{A}_n = (ig)^{n-2} \sum_{\sigma(2,3,\ldots,n-1)} \widetilde{f}^{a_1 a_{\sigma(2)} x_1}\widetilde{f}^{x_1 a_{\sigma(3)} x_2}\cdots\widetilde{f}^{x_{n-3} a_{\sigma(n-1)} a_n} A_n(1,\sigma(2),\ldots,\sigma(n-1),n)\,. \tag{2.13}$$

This form can be obtained by using the Jacobi identity $f^{dac}f^{cbe} - f^{dbc}f^{cae} = f^{abc}f^{dce}$ to bring all contractions between f^{abc} structure constants, from a generic Feynman diagram, into the form $\widetilde{f}^{a_1 a_{\sigma(2)} x_1}\widetilde{f}^{x_1 a_{\sigma(3)} x_2}\cdots\widetilde{f}^{x_{n-3} a_{\sigma(n-1)} a_n}$. Notice the smaller permutation sum compared to eq. (2.12). We will return to the connection between these two color decompositions below.

Let us note that we can transform an incoming particle into an outgoing antiparticle with the opposite helicity (or vice versa). Using this we will always take *all* particles to be *outgoing*. This in particular implies that conservation of momentum takes the form $\sum_i p_i = 0$.

Properties of Subamplitudes

Naively it seems like we have not gained much by introducing the above decomposition. Indeed eq. (2.12) contains $(n-1)!$ subamplitudes A_n which we would in principle need to calculate. However, the color-ordered subamplitudes are a lot simpler than the full amplitude, and satisfy several useful properties which make calculations considerably easier. Often they also reveal underlying structures that would have otherwise been hidden from us. Some of the better known of these features follow from the structure of the gauge group itself and will be briefly reviewed here.

First of all the subamplitudes are gauge-invariant quantities. This can be seen from the observation that for two permutations $\{a\}$ and $\{b\}$ of the color indices we have

$$\sum_{i=1}^{n}\sum_{a_i=1}^{N^2-1} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]\big(\mathrm{Tr}[T^{b_1}T^{b_2}\cdots T^{b_n}]\big)^* = N^{n-2}(N^2-1)(\delta_{\{a\}\{b\}} + \mathcal{O}(N^{-2}))\,, \tag{2.14}$$

and since gauge invariance must hold order by order in $1/N$, each A_n has to be gauge invariant [2]. The gauge invariance of subamplitudes implies the freedom to choose different gauges in the calculation of each subamplitude thereby allowing us to always pick the one that simplifies calculations most. Let us also note that it is this "orthogonality" which can be used to show that the subamplitudes in eq. (2.13) are the same as those in eq. (2.12). In other words, the kinematic coefficients of the $\widetilde{f}^{a_1 a_{\sigma(2)} x_1}\widetilde{f}^{x_1 a_{\sigma(3)} x_2}\cdots\widetilde{f}^{x_{n-3} a_{\sigma(n-1)} a_n}$ color factors in eq. (2.13) are the same as the coefficients to the $\mathrm{Tr}[T^{a_1}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n-1)}}T^{a_n}]$ color factors in eq. (2.12).

Although clever gauge choices can reduce the amount of calculations considerably, a more powerful feature are the relations that exist among the subamplitudes, reducing the number of independent subamplitudes significantly.

The two most obvious relations are the invariance under cyclic permutations and reflection symmetry of legs

$$\begin{aligned} A_n(1,2,\ldots,n) &= A_n(2,3,\ldots,n,1)\,,\\ A_n(n,n-1,\ldots,1) &= (-1)^n A_n(1,2,\ldots,n)\,, \end{aligned} \tag{2.15}$$

which follow from the trace structure when performing the color decomposition in eq. (2.12).

Another important relation is the "photon"-decoupling identity

$$\sum_{\sigma\in cyclic} A_n(1,\sigma(2,3,\ldots,n)) = 0\,, \tag{2.16}$$

where the sum runs over all cyclic permutations of leg $2,3,\ldots,n$, *e.g.*

$$A_4(1,2,3,4) + A_4(1,3,4,2) + A_4(1,4,2,3) = 0\,. \tag{2.17}$$

The name comes from the fact that the full tree amplitudes of the $U(N)$ gauge theory can be decomposed similarly to eq. (2.12), and the additional generator, compared to $SU(N)$, is proportional to the identity matrix. Since the identity matrix commutes with all other generators, the related field does not interact with any of the gluons and behaves more like a photon. Any amplitude containing this extra photon must therefore vanish. Substituting one of the generators in eq. (2.12) with the identity matrix and collecting terms with identical color factors leads to equations like (2.16).

More generally the subamplitudes satisfy the Kleiss-Kuijf (KK) relations [1,13]

$$A_n(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in \mathrm{OP}(\{\alpha\},\{\beta^T\})} A_n(1,\sigma,n)\,, \tag{2.18}$$

with the sum running over "ordered permutations", *i.e.* all permutations of $\{\alpha\}\cup\{\beta^T\}$ that keep the order of the elements belonging to each set. n_β is the number of elements in $\{\beta\}$, and $\{\beta^T\}$ is the $\{\beta\}$ set with the ordering reversed.

As an example, consider $A_6(1,\{2,3\},6,\{4,5\})$. Here $\{\alpha\} = \{2,3\}$, $\{\beta\} = \{4,5\}$ and $n_\beta = 2$, hence

$$\begin{aligned} A_6(1,\{2,3\},6,\{4,5\}) = {} & A_6(1,2,3,5,4,6) + A_6(1,2,5,3,4,6) + A_6(1,2,5,4,3,6) \\ & + A_6(1,5,2,4,3,6) + A_6(1,5,4,2,3,6) + A_6(1,5,2,3,4,6)\,. \end{aligned} \tag{2.19}$$

The KK-relations include the reflection symmetry, take $\{\alpha\} = \emptyset$ and $\{\beta\} = \{2,\ldots,n-1\}$, as well as the photon-decoupling identities, take the $\{\beta\}$ set to contain only one leg and have the rest in the $\{\alpha\}$ set, as special cases. They also exactly relate the two different color decompositions in eq. (2.12) and (2.13) [13]. Once the two decompositions have been established the KK-relations can be proven directly by using

$$\widetilde{f}^{a_1 a_2 x_1}\widetilde{f}^{x_1 a_3 x_2}\cdots\widetilde{f}^{x_{n-3} a_{n-1} a_n} = (-i)^{n-2}\mathrm{Tr}(T^{a_1}[T^{a_2},[T^{a_3},\ldots,[T^{a_{n-1}},T^{a_n}]\ldots]])\,, \tag{2.20}$$

in eq. (2.13) and identifying all terms contributing to the trace

$$\mathrm{Tr}(T^{a_1}T^{a_{\alpha(1)}}\cdots T^{a_{\alpha(n-2-q)}}T^{a_n}T^{a_{\beta(1)}}\cdots T^{a_{\beta(q)}})\,. \tag{2.21}$$

As we see from eq. (2.18), or the color decomposition in eq. (2.13), the number of independent subamplitudes is reduced to $(n-2)!$. However, in the next chapter we will see that we can do even better, and reduce this number down to $(n-3)!$.

After this short introduction to Yang-Mills theory with $SU(N)$ gauge group let us briefly turn to the second field theory examined in this thesis, perturbative gravity.

2.2 Perturbative Gravity

The Lagrangian for pure Einstein gravity is given by

$$\mathcal{L}_{EG} = \frac{2}{\kappa^2}\sqrt{-g}R\,, \tag{2.22}$$

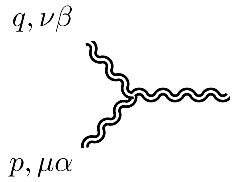
where $g \equiv \det g_{\mu\nu}$, R is the scalar curvature and κ is the gravitational coupling constant related to Newton's constant G by $\kappa^2 = 32\pi G$.

As an attempt to formulate this as a quantum field theory, we take the naive approach of perturbatively expanding the Lagrangian, *i.e.* considering a small deviation $h_{\mu\nu}$ from the flat spacetime metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,. \tag{2.23}$$

Here $h_{\mu\nu}$ is to be identified as the graviton field. Expanding out g and R in powers of κ in eq. (2.22) one gets expressions for the graviton propagator and the interaction terms. Whereas Yang-Mills theory only contains three- and four-point vertices the expansion in κ leads to an infinite number of more and more complicated vertices for gravitons. As examples we present the propagator (in de Donder gauge) and the three-point vertex [31,32]:

$\alpha\beta$  $\gamma\delta \;=\; -i\dfrac{\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma}-\eta_{\alpha\beta}\eta_{\gamma\delta}}{2p^2}\,,$ (2.24)



$$
\begin{aligned}
k, \sigma\gamma = \mathrm{sym}\,\big[& -\tfrac{1}{2}P_3(p\cdot q\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(p_\nu p_\beta\,\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\
& + \tfrac{1}{2}P_3(p\cdot q\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(p\cdot q\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) \\
& + 2P_3(p_\nu p_\gamma\,\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(p_\beta q_\mu\,\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
& + P_3(p_\sigma q_\gamma\,\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(p_\sigma p_\gamma\,\eta_{\mu\nu}\eta_{\alpha\beta}) \\
& + 2P_6(p_\nu q_\gamma\,\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(p_\nu q_\mu\,\eta_{\beta\sigma}\eta_{\gamma\alpha}) \\
& - 2P_3(p\cdot q\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\big]\;,
\end{aligned}
\tag{2.25}
$$

where sym$[\cdots]$ means that each pair of indices, *i.e.* $\mu\alpha, \nu\beta, \sigma\gamma$, is to be symmetrized and P means that we should sum over all distinct permutations of the momenta. The subscripts on the P's denote the number of terms in the sum.

We see that the three-point vertex contains 45 terms which then have to be symmetrized in the pairs of indices. If we go to the four-point vertex we will get about 360 terms which then have to be symmetrized, and the complications will only get worse as we continue. With this in mind, and the fact that a n-point graviton tree-level amplitude will involve up to n-point vertices, it seems very compelling to conclude that gravity is far more complicated than Yang-Mills theory. Any connection between these theories seems to almost require a miracle.

Much of the apparent mess one finds in the Feynman diagram method comes from propagators and vertices involving unphysical (off-shell) gauge-dependent states. As we will see, remarkable simplifications start to occur when we go on-shell.

At this point we should stress that the above quantum field theory of gravity is nonrenormalizable. It is ultraviolet divergent at two loops [33, 34], but in general non-supersymmetric gravity coupled to matter will be divergent already at one loop [35]. This does not mean that we should necessarily discard these theories completely, but rather think of them as effective field theories of gravity only valid for low energies. In most of what follows we will only consider tree-level amplitudes and the nonrenormalizability is therefore of no concern to us. Later we will comment on supersymmetric gravity which seems to be more well behaved.

It is hard to exaggerate the importance of using good variables when looking for simplifying structures and patterns in explicit expressions. One of the most successful languages used in the field of scattering amplitudes has been the so-called *spinor helicity formalism.* This will be the topic of the next section.

2.3 Spinor Helicity Formalism

The basic idea of the spinor helicity formalism is to reduce all mathematical objects used in expressing scattering amplitudes (*e.g.* Dirac spinors, polarization vectors and momenta) to objects belonging to the two-dimensional irreducible representations of the Lorentz group. Recall that the Lie algebra of the Lorentz group can be expressed as the Lie algebra of $SU(2)_L \times SU(2)_R$. The finite-dimensional representations of the Lorentz group can therefore be classified as (j_L, j_R), where j_L and j_R are integers or half-integers. The two two-dimensional representations we will use as building blocks are $(1/2, 0)$ and $(0, 1/2)$. These are fundamental representations in the sense that higher finite-dimensional representations of the Lorentz group can be constructed from them.

As usual we will talk about a spinor/vector *belonging* to a certain representation, although strictly speaking they belong to the space on which the actual Lorentz transformations in the representation act.

The two-component spinors transforming in $(1/2,0)$ and $(0,1/2)$ are denoted

$$\lambda_a\,, \qquad \tilde{\lambda}^{\dot{a}}\,, \tag{2.26}$$

respectively, and the spinor indices are raised and lowered by the antisymmetric 2×2 matrix

$$\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.27}$$

i.e.

$$\lambda^a = \lambda_b\epsilon^{ba}, \qquad \lambda_a = \epsilon_{ab}\lambda^b, \qquad \tilde{\lambda}_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}^{\dot{b}}, \qquad \tilde{\lambda}^{\dot{a}} = \tilde{\lambda}_{\dot{b}}\epsilon^{\dot{b}\dot{a}}\,. \tag{2.28}$$

We can define the following Lorentz invariant spinor products

$$\langle\lambda\eta\rangle \equiv \epsilon^{ab}\lambda_a\eta_b = \lambda^a\eta_a = \lambda_1\eta_2 - \lambda_2\eta_1\,, \tag{2.29}$$

$$[\tilde{\lambda}\tilde{\eta}] \equiv \epsilon_{\dot{b}\dot{a}}\tilde{\lambda}^{\dot{a}}\tilde{\eta}^{\dot{b}} = \tilde{\lambda}_{\dot{a}}\tilde{\eta}^{\dot{a}} = \tilde{\lambda}^{\dot{2}}\tilde{\eta}^{\dot{1}} - \tilde{\lambda}^{\dot{1}}\tilde{\eta}^{\dot{2}}\,, \tag{2.30}$$

which are obviously antisymmetric

$$\langle\lambda\eta\rangle = -\langle\eta\lambda\rangle\,, \qquad [\tilde{\lambda}\tilde{\eta}] = -[\tilde{\eta}\tilde{\lambda}]\,, \tag{2.31}$$

and hence $\langle\lambda\lambda\rangle = 0$ and $[\tilde{\lambda}\tilde{\lambda}] = 0$.

With this $\langle\cdot\rangle$ and $[\cdot]$ notation of the spinor products we will often just write the spinors as

$$\lambda^a = \langle\lambda|\,, \qquad \lambda_a = |\lambda\rangle\,, \qquad \tilde{\lambda}_{\dot{a}} = [\lambda|\,, \qquad \tilde{\lambda}^{\dot{a}} = |\lambda]\,. \tag{2.32}$$

Four-vectors P^μ belong to the $(1/2,1/2)$ representation of the Lorentz group and can be mapped to a bi-spinor through the 2×2 Pauli matrices $\boldsymbol{\sigma} = (\sigma^1,\sigma^2,\sigma^3)$, and the identity matrix ($\sigma^0 = 1$)

$$P_{\dot{a}b} \equiv \sigma_{\mu,\dot{a}b}P^\mu = \begin{pmatrix} P^0 - P^3 & -P^1 - iP^2 \\ -P^1 + iP^2 & P^0 + P^3 \end{pmatrix}, \tag{2.33}$$

where

$$\sigma^\mu_{a\dot{b}} = (1,\boldsymbol{\sigma}), \qquad \sigma^{\mu,\dot{a}b} = (1,-\boldsymbol{\sigma})\,. \tag{2.34}$$

The determinant of the bi-spinor is precisely the square of the corresponding four-vector $\det(P_{\dot{a}b}) = P^2$. When the determinant is zero (*i.e.* the four-vector is lightlike) the rank of the 2×2 matrix is one and it can be decomposed into a single spinor (outer) product

$$P_{\dot{a}b} = \tilde{p}_{\dot{a}}p_b = |p\rangle[p|\,, \tag{2.35}$$

where $\tilde{p}_{\dot{a}}$ and p_b are two-component spinors belonging to $(0,1/2)$ and $(1/2,0)$, respectively. Notice that the decomposition in eq. (2.35) is not unique since $P_{\dot{a}b}$ is invariant under the following little group transformation

$$p_b \quad\longrightarrow\quad tp_b\,, \qquad \tilde{p}_{\dot{a}} \quad\longrightarrow\quad t^{-1}\tilde{p}_{\dot{a}}\,. \tag{2.36}$$

When the four-vectors represent real momenta in Minkowski space the two spinors are related by complex conjugation $(\tilde{p})^* = \pm p$ (which would also constrain the freedom above to t being a phase factor). However, we will often take them to be independent which correspond to working with complex momenta.

Furthermore, the spinor products in eq. (2.29) and (2.30) are nicely related to the usual vector product through

$$\langle kp\rangle[pk] = \tilde{p}_{\dot{a}} p_b \tilde{k}^{\dot{a}} k^b = P_{\dot{a}b} K^{\dot{a}b} = P_\mu \sigma^\mu_{\dot{a}b} \sigma^{\nu,\dot{a}b} K_\nu = P_\mu 2 g^{\mu\nu} K_\nu = 2P \cdot K \,. \tag{2.37}$$

Beside the momenta, scattering amplitudes also depend on the helicities of external states, which, for spin-1 vector bosons, are represented by a polarization vector $\varepsilon^\pm_\mu$. If we are only given the momentum and helicity (*i.e.* $+$ or $-$) of a particle there is actually no natural way to pick $\varepsilon^\pm_\mu$ [4]. However, once we have chosen a decomposition of the momentum in terms of spinors we can immediately write down the following bi-spinors for the polarization vectors

$$\varepsilon^-_{\dot{a}b}(p,q) = \frac{\sqrt{2} q_{\dot{a}} p_b}{[pq]} \,, \qquad \varepsilon^+_{\dot{a}b}(p,q) = \frac{\sqrt{2} p_{\dot{a}} q_b}{\langle qp\rangle} \,, \tag{2.38}$$

where p is the momentum of the particle, and q an (almost) arbitrary *reference spinor*, reflecting the freedom of on-shell gauge transformations. The polarization vectors are normalized such that

$$\varepsilon^+ \cdot \varepsilon^- = \frac{1}{2} \varepsilon^+_{\dot{a}b} \varepsilon^{-,\dot{a}b} = \frac{2 p_{\dot{a}} q_b q^{\dot{a}} p^b}{2\langle qp\rangle[pq]} = \frac{\langle pq\rangle[pq]}{\langle qp\rangle[pq]} = -1 \,. \tag{2.39}$$

Under a little group transformation on the q spinors the polarization vectors do not change, but on the momentum spinors the polarization vectors transform as

$$\varepsilon^+ \quad \longrightarrow \quad t^{-2}\varepsilon^+ \,, \qquad \varepsilon^- \quad \longrightarrow \quad t^2 \varepsilon^- \,. \tag{2.40}$$

Finally, for the spin-2 gravitons the polarization is described by a polarization tensor $\epsilon^\pm_{\mu\nu}$, which can be written in terms of the above polarization vectors

$$\epsilon^\pm_{\mu\nu} = \varepsilon^\pm_\mu \varepsilon^\pm_\nu \,. \tag{2.41}$$

Helicity Amplitudes

Since it will be an important starting point for several of the later derivations, and since it is not always spelled out very explicitly in the literature, let us start by spending some time on the special case of three-point amplitudes.

Three-Point Amplitudes

Formally the color-ordered three-point amplitude of Yang-Mills theory is given by the three-point vertex in eq. (2.9) (stripping off the f^{abc} color factor) contracted with the polarization vectors of external states, *e.g.* for helicity $(--+)$

$$A_3(1^-,2^-,3^+) \sim \varepsilon^-_1 \cdot \varepsilon^-_2 (k_1 - k_2) \cdot \varepsilon^+_3 + \varepsilon^-_2 \cdot \varepsilon^+_3 (k_2 - k_3) \cdot \varepsilon^-_1 + \varepsilon^-_1 \cdot \varepsilon^+_3 (k_3 - k_1) \cdot \varepsilon^-_2 \,. \tag{2.42}$$

From momentum conservation $p_1 + p_2 + p_3 = 0$ and the on-shell condition $p_i^2 = 0$ it follows that $p_i \cdot p_j = 0$ for any pair of $i,j = 1,2,3$. In terms of the spinors this is

$$\langle 12\rangle[21] = 0 \,, \qquad \langle 23\rangle[32] = 0 \,, \qquad \langle 13\rangle[31] = 0 \,. \tag{2.43}$$

If $[21] = 0$ and $[32] = 0$, then we also have $[31] = 0$, because if $|1]$ and $|3]$ are both proportional to $|2]$ they must also be proportional to each other. As mentioned earlier, if we work in Minkowski space with real momenta we have $|i\rangle^* \sim |i]$, so also $\langle 12\rangle = \langle 23\rangle = \langle 13\rangle = 0$. It then follows that

$$k_i \cdot \varepsilon_j^- \propto \langle ij\rangle = 0\,, \qquad k_i \cdot \varepsilon_j^+ \propto [ij] = 0\,, \tag{2.44}$$

and we immediately see from eq. (2.42) that $A_3(1^-, 2^-, 3^+) = 0$.

However, if we consider complex momenta, then $|i\rangle$ and $|i]$ are independent and we can satisfy both momentum conservation and the on-shell condition with $[ij] = 0$, but still have $\langle ij\rangle \neq 0$. We take the polarization vectors as

$$\varepsilon^-_{1,\dot{a}b}(p_1, q) = \frac{\sqrt{2}q_{\dot{a}}p_{1b}}{[1q]}\,, \qquad \varepsilon^-_{2,\dot{a}b}(p_2, q) = \frac{\sqrt{2}q_{\dot{a}}p_{2b}}{[2q]}\,, \qquad \varepsilon^+_{3,\dot{a}b}(p_3, p_1) = \frac{\sqrt{2}p_{3\dot{a}}p_{1b}}{\langle 13\rangle}\,, \tag{2.45}$$

where we have put $q_1 = q_2 = q$ and $q_3 = p_1$. Note that we can not set q to be any of the external states since this would give a zero in the denominator, but since $\langle ij\rangle \neq 0$ this is not a problem in the ε_3^+ state (except for $q_3 = p_3$ of course) where we choose the reference momentum to equal p_1. It is easy to see that both the first and the last term of eq. (2.42) are zero, so we are left with

$$\begin{aligned} A_3(1^-, 2^-, 3^+) &\sim \varepsilon_2^- \cdot \varepsilon_3^+ (k_2 - k_3) \cdot \varepsilon_1^- = 2(\varepsilon_2^- \cdot \varepsilon_3^+)(k_2 \cdot \varepsilon_1^-) \\ &= 2\left(\frac{\langle 12\rangle[q3]}{\langle 13\rangle[2q]}\right)\left(\frac{\langle 12\rangle[2q]}{\sqrt{2}[1q]}\right) \\ &= \sqrt{2}\frac{\langle 12\rangle^2[q3]}{\langle 31\rangle[q1]}\,. \end{aligned} \tag{2.46}$$

Using momentum conservation

$$\langle 2|\Big(|1\rangle[1| + |2\rangle[2| + |3\rangle[3|\Big)|q] = 0 \quad \Longrightarrow \quad \frac{[q3]}{[q1]} = \frac{\langle 12\rangle}{\langle 23\rangle}\,, \tag{2.47}$$

we end up with

$$A_3(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,. \tag{2.48}$$

In a completely similar way we can consider the solution for eq. (2.43) with $\langle ij\rangle = 0$ and $[ij] \neq 0$ instead, and find the following non-zero three-point amplitude

$$A_3(1^+, 2^+, 3^-) = \frac{[12]^3}{[23][31]}\,. \tag{2.49}$$

Now let us look at the graviton three-point amplitude in complex momenta. Contracting the three-point vertex in eq. (2.25) with the polarization tensors $\epsilon_1^{-,\mu\alpha}$, $\epsilon_2^{-,\nu\beta}$ and $\epsilon_3^{+,\sigma\gamma}$, and using $k_i \cdot k_j = 0$ it reduces to [32]

$$\begin{aligned} M_3(1^-, 2^-, 3^+) \sim\ & (k_1\epsilon_3^+k_1)(\epsilon_1^-\epsilon_2^-) + (k_3\epsilon_2^-k_2)(\epsilon_1^-\epsilon_3^+) + (k_2\epsilon_1^-k_2)(\epsilon_2^-\epsilon_3^+) \\ & + 2(k_2\epsilon_1^-\epsilon_2^-\epsilon_3^+k_1) + 2(k_3\epsilon_2^-\epsilon_1^-\epsilon_3^+k_1) + 2(k_2\epsilon_1^-\epsilon_3^+\epsilon_2^-k_3)\,. \end{aligned} \tag{2.50}$$

Using eq. (2.41) this is

$$\begin{aligned} M_3(1^-, 2^-, 3^+) \sim\ & (k_1 \cdot \varepsilon_3^+)^2(\varepsilon_1^- \cdot \varepsilon_2^-)^2 + (k_3 \cdot \varepsilon_2^-)^2(\varepsilon_1^- \cdot \varepsilon_3^+)^2 + (k_2 \cdot \varepsilon_1^-)^2(\varepsilon_2^- \cdot \varepsilon_3^+)^2 \\ & + 2(k_2 \cdot \varepsilon_1^-)(\varepsilon_1^- \cdot \varepsilon_2^-)(\varepsilon_2^- \cdot \varepsilon_3^+)(\varepsilon_3^+ \cdot k_1) \\ & + 2(k_3 \cdot \varepsilon_2^-)(\varepsilon_2^- \cdot \varepsilon_1^-)(\varepsilon_1^- \cdot \varepsilon_3^+)(\varepsilon_3^+ \cdot k_1) \\ & + 2(k_2 \cdot \varepsilon_1^-)(\varepsilon_1^- \cdot \varepsilon_3^+)(\varepsilon_3^+ \cdot \varepsilon_2^-)(\varepsilon_2^- \cdot k_3)\,, \end{aligned} \tag{2.51}$$

and with the polarization vectors from eq. (2.45), which satisfy

$$k_1 \cdot \varepsilon_3^+ = \varepsilon_1^- \cdot \varepsilon_3^+ = \varepsilon_1^- \cdot \varepsilon_2^- = 0\,, \tag{2.52}$$

we see that only the third term survives, *i.e.*

$$\begin{aligned} M_3(1^-, 2^-, 3^+) &\sim (k_2 \cdot \varepsilon_1^-)^2 (\varepsilon_2^- \cdot \varepsilon_3^+)^2 \\ &= \left[(k_2 \cdot \varepsilon_1^-)(\varepsilon_2^- \cdot \varepsilon_3^+)\right]^2 . \end{aligned} \tag{2.53}$$

Comparing to the above Yang-Mills three-point calculation this is

$$M_3(1^-, 2^-, 3^+) = \left[A_3(1^-, 2^-, 3^+)\right]^2 = \frac{\langle 12\rangle^6}{\langle 23\rangle^2 \langle 31\rangle^2}\,. \tag{2.54}$$

In a similar manner we can also get

$$M_3(1^+, 2^+, 3^-) = \left[A_3(1^+, 2^+, 3^-)\right]^2 = \frac{[12]^6}{[23]^2[31]^2}\,, \tag{2.55}$$

if we work with the other solution to eq. (2.43).

Finally let us note that even for complex momenta we will always have $M_3(1^\pm, 2^\pm, 3^\pm) = A_3(1^\pm, 2^\pm, 3^\pm) = 0$.

In this section we worked out the three-point amplitudes directly from the Feynman rules. Alternatively they can also be uniquely determined from more general considerations; mainly the constraints from assuming Lorentz invariance and an auxiliary condition for helicity amplitudes reflecting the same kind of (helicity-dependent) scaling as we saw in eq. (2.40) [36]. The result is of course the same as obtained above.

Higher-Point Amplitudes

For scattering amplitudes of $n \geq 4$ external legs with all helicities or all but one being the same the amplitude vanishes. This can be shown either using recursion relations or by supersymmetry [37,38]. The first non-vanishing case is the one with two equal helicity legs and the rest of opposite helicity. These are known as Maximally-Helicity-Violating (MHV) amplitudes. In the spinor formalism they take a very simple form. For the pure gluon n-point amplitude they are given by the famous Parke-Taylor formula [3]

$$A_n^{MHV} = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}\,, \tag{2.56}$$

where i, j denote the two negative helicity gluons. Likewise we can write the conjugate amplitude

$$A_n^{MHV} = (-1)^n \frac{[ij]^4}{[12][23]\cdots[n1]}\,, \tag{2.57}$$

now with i, j denoting the two positive helicity gluons.

For pure graviton MHV amplitudes the expressions are more complicated. There are several different looking, but equivalent, n-point formulas, for instance [39]

$$M_n^{MHV} = \frac{\langle ij\rangle^8}{\langle n-1|n\rangle^2} \left(\prod_{a=1}^{n-2} \frac{1}{(\langle a|n-1\rangle\langle a|n\rangle)^2}\right) \sum_{trees} \prod_{edges\ ab} \frac{[ab]}{\langle ab\rangle} \langle a|n-1\rangle\langle b|n-1\rangle\langle an\rangle\langle bn\rangle\,, \tag{2.58}$$

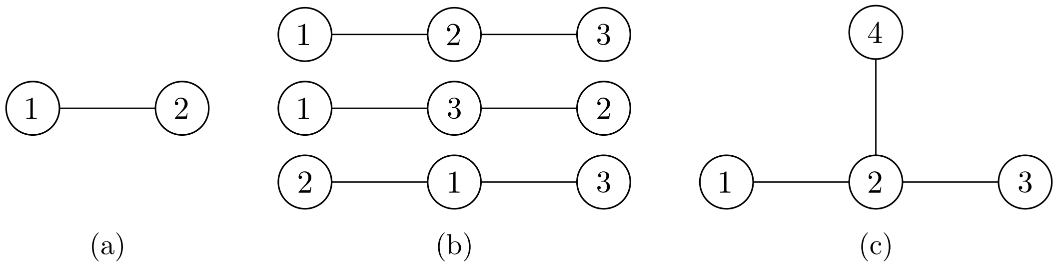


Figure 2.1: Tree graphs representing terms in eq. (2.58). (a) The only graph at four points. (b) The three graphs at five points. (c) Example of a "non-flat" graph in the six point case.

where the sum runs over all inequivalent connected tree graphs with vertices labeled $1, 2, \ldots, n-2$, and i, j are the negative helicity legs.

For example, at four points there is only one tree graph, the graph with one edge connecting 1 and 2, see (a) in figure 2.1, and eq. (2.58) gives

$$M_4^{MHV} = \frac{\langle ij\rangle^8[12]}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2} \,. \tag{2.59}$$

At five points there are the three tree graphs shown in (b) in figure 2.1. The first graph gives

$$\frac{\langle ij\rangle^8[12][23]}{\langle 12\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle^2} \,, \tag{2.60}$$

with the two remaining graphs having very similar looking expressions. Adding them all up gives the five-point MHV graviton amplitude. Going beyond five points we begin to get "non-flat" graphs as also illustrated in figure 2.1. In general there are $(n-2)^{n-4}$ tree graphs to add up.

The expressions get more complicated as we go to non-MHV amplitudes, *e.g.* (six-gluon NMHV amplitude)

$$\begin{aligned} A_6(1^+,2^+,3^+,4^-,5^-,6^-) = & \frac{\langle 6|1+2|3]^3}{\langle 12\rangle\langle 61\rangle[34][45]\langle 2|1+6|5]s_{126}} \\ & + \frac{\langle 4|5+6|1]^3}{\langle 23\rangle\langle 34\rangle[56][61]\langle 2|1+6|5]s_{156}} \,, \end{aligned} \tag{2.61}$$

where $s_{ijk} \equiv (p_i + p_j + p_k)^2$ and $\langle i|k+m|j] \equiv \langle ik\rangle[kj] + \langle im\rangle[mj]$. Considering the enormous amount of Feynman diagrams that should be added to obtain this expression, it is still remarkably simple.

We have several times pointed out how complicated and time consuming amplitude calculations can get as we increase the number of external legs. Luckily, it turns out that there exist very powerful tools with which we can "recycle" our work as we go from n to $n+1$ external legs. These are known as recursion relations, and will be the topic of the next section.

2.4 Recursion Relations

There are three different recursion relations known for general n-point tree amplitudes. The oldest one is the Berends-Giele recursion from 1987 [40]. It is based on the introduction of an off-shell current, from which color-ordered amplitudes can be inferred. The recursion relation itself gives the n-point current in terms of lower-point currents and the three- and four-point gluon vertices. This was, for instance, used to give the first rigorous proof of the above Parke-Taylor formula [40]. As indicated in the introduction, after Witten's famous twistor string paper, the recursive structure of amplitudes started to become more clear. This inspired the CSW formalism [5, 6], using an off-shell continuation of (lower-point) MHV amplitudes to calculate higher-point (not necessarily MHV) color-ordered amplitudes. Shortly after came the BCFW relations, also constructing higher-point amplitudes, but this time from *on-shell* lower-point ones [7,8]. This is the recursion relation we will be using in this thesis. More precisely, we will be using the same method from which it can be derived [8]. We will therefore here spend a little time on the details of the derivation.

BCFW Recursion

Start by deforming two of the external momenta, say p_j and p_l, as

$$\begin{aligned} p_j &\longrightarrow \widehat{p}_j = p_j - zq\,, \\ p_l &\longrightarrow \widehat{p}_l = p_l + zq\,, \end{aligned} \tag{2.62}$$

where z is a complex variable and q is a four-vector satisfying $q^2 = q \cdot p_j = q \cdot p_l = 0$. This preserves conservation of momentum and the on-shell condition. At tree-level A_n is a rational function of the external momenta, implying that the deformed amplitude $A_n(z)$ is a rational function of z. In the $z \to \infty$ limit $A_n(z)$ goes to zero, so Cauchy's Theorem tells us that

$$0 = \frac{1}{2\pi i} \oint_\infty \frac{dz}{z} A_n(z) = A_n(0) + \sum_{\text{poles } z_p \neq 0} \frac{\text{Res}_p(A_n(z), z_p)}{z_p}\,, \tag{2.63}$$

where $A_n(0)$ is the undeformed amplitude, coming from the residue of the $z = 0$ pole. The remaining poles (those different from $z = 0$) come from $A_n(z)$, and can only originate from Feynman propagators going on-shell, *i.e.* when $P_{k,m}(z)^2$ vanishes, where

$$P_{k,m}(z) \equiv p_k + p_{k+1} + \cdots + \widehat{p}_j + \cdots + p_m, \tag{2.64}$$

with $j \in \{k, \ldots, m\}$ and $l \notin \{k, \ldots, m\}$ (or vice versa). We do not get poles in z if $j, l \in \{k, \ldots, m\}$ since such a $P_{k,m}$ is independent of z.

We see that

$$P_{k,m}(z)^2 = P_{k,m}^2 - 2zq \cdot P_{k,m}\,, \tag{2.65}$$

where $P_{k,m} = P_{k,m}(0)$, and hence

$$z = -\frac{P_{k,m}(z)^2}{2(q \cdot P_{k,m})} + \frac{P_{k,m}^2}{2(q \cdot P_{k,m})}\,. \tag{2.66}$$

Let us denote the value of z where $P_{k,m}(z)$ is going on-shell by $z_{k,m}$, *i.e.*

$$z_{k,m} = \frac{P_{k,m}^2}{2(q \cdot P_{k,m})}\,. \tag{2.67}$$

The poles of $A_n(z)$ are *simple poles*, so the residues can be calculated as

$$\text{Res}_p(A_n(z), z_{k,m}) = \lim_{z \to z_{k,m}} [(z - z_{k,m}) A_n(z)] = -\frac{\lim_{z \to z_{k,m}} [P_{k,m}(z)^2 A_n(z)]}{2(q \cdot P_{k,m})}\,, \tag{2.68}$$

from which we get

$$\frac{\mathrm{Res}_p(A_n(z), z_{k,m})}{z_{k,m}} = -\frac{\lim_{z\to z_{k,m}}[P_{k,m}(z)^2 A_n(z)]}{P_{k,m}^2}\,. \tag{2.69}$$

Using the general factorization property of gluon amplitudes

$$A_n \xrightarrow{P^2\to 0} \sum_{h=\pm} A_{r+1}(k,\ldots,m,-P^{-h})\frac{1}{P^2}A_{n-r+1}(P^h, m+1,\ldots,k-1)\,, \tag{2.70}$$

we see that

$$\frac{\mathrm{Res}_p(A_n(z), z_{k,m})}{z_{k,m}} = -\sum_{h=\pm} A_{r+1}(k,\ldots,m,-\widehat{P}_{k,m}^{-h})\frac{1}{P_{k,m}^2}A_{n-r+1}(\widehat{P}_{k,m}^h, m+1,\ldots,k-1)\,, \tag{2.71}$$

with $\widehat{P}_{k,m} \equiv P_{k,m}(z_{k,m})$. Hence, combined with eq. (2.63), we find the BCFW recursion relation [7, 8]

$$A_n = \sum_r \sum_{h=\pm} A_{r+1}(k,\ldots,m,-\widehat{P}_{k,m}^{-h})\frac{1}{P_{k,m}^2}A_{n-r+1}(\widehat{P}_{k,m}^h, m+1,\ldots,k-1)\,, \tag{2.72}$$

where the sum, denoted with r, runs over all different contributions from deformed propagators going on-shell.

This derivation was done in the context of color-ordered tree amplitudes, but the recursion relation is also valid for tree-level gravity amplitudes [9]. The only change will be in the sum over contributing terms, *i.e.* in the r sum above. Since gravity amplitudes are totally permutation invariant we need to include *all* different combinations of momenta where the pole includes one of the deformed legs . For instance, making a BCFW-shift in leg 1 and 4, the four-point color-ordered gluon amplitude $A_4(1^-,2^-,3^+,4^+)$ can be represented as

$$A_4(1^-,2^-,3^+,4^+) = \sum_{h=\pm} A_3(\widehat{1}^-,2^-,-\widehat{P}_{12}^{-h})\frac{1}{s_{12}}A_3(\widehat{P}_{12}^h,3^+,\widehat{4}^+)\,, \tag{2.73}$$

whereas the four-point graviton amplitude is

$$\begin{aligned} M_4(1^-,2^-,3^+,4^+) = &\sum_{h=\pm} M_3(\widehat{1}^-,2^-,-\widehat{P}_{12}^{-h})\frac{1}{s_{12}}M_3(\widehat{P}_{12}^h,3^+,\widehat{4}^+) \\ &+\sum_{h=\pm} M_3(\widehat{1}^-,3^+,-\widehat{P}_{13}^{-h})\frac{1}{s_{13}}M_3(\widehat{P}_{13}^h,2^-,\widehat{4}^+)\,. \end{aligned} \tag{2.74}$$

It has been shown that the fall-off at $z\to\infty$ for the graviton amplitude is even stronger than one would naively have guessed [9, 10, 41–49].

With the BCFW recursion relations and the three-point amplitudes from section 2.3, we can in principle calculate any n-point amplitude. In this sense the three-point amplitudes are building blocks for any higher-point amplitude. A natural question then arises; what are the consequences of the squaring relations in eq. (2.54) and (2.55) for higher points? Just plugging the squaring relations directly into expressions like eq. (2.74) does not reveal much general structure, but we will see how to address this question soon.

2.5 Superamplitudes

Many of the properties we will consider in this thesis can be directly generalized to supersymmetric theories. We will even see that some of them are more natural explained in the framework of supersymmetry. Since the particle content of most supersymmetric theories is rather large, it is convenient to introduce a formalism that neatly encodes the full spectrum into well-behaved objects. This can be achieved with the *superfield formalism*, which will be reviewed below.

Superfield Formalism in Maximally Supersymmetric Theories

Supersymmetry can be thought of as an extension of the Poincaré algebra with $\mathcal{N}$ additional anticommuting spin-1/2 operators Q, called supercharges. Since these operators are fermionic their action on a state will change its statistics, *i.e.* supersymmetry transforms fermions to bosons and vice versa.

In $\mathcal{N}=4$ super-Yang-Mills theory the supercharges $Q^a = (Q^a_\alpha, \widetilde{Q}^a_{\dot\alpha})$ are labeled by the $SU(4)_R$ R-symmetry index $a = 1, \ldots, 4$. This represents a (complex) rotational symmetry between the supercharges. The $\alpha, \dot\alpha = 1, 2$ are spinor indices.

The 16 states of $\mathcal{N}=4$ super-Yang-Mills theory are given by 2 gluons g_+, g_-^{1234}, 4 pairs of gluinos f_+^a, f_-^{abc} and 6 scalars s^{ab} satisfying the reality condition $s^{ab} = \epsilon^{abcd} s_{cd}/2$, where $s^{ab} \equiv s^\dagger_{ab}$. They transform under the $SU(4)_R$ R-symmetry as antisymmetric products in the fundamental representation and therefore carry antisymmetric fundamental $SU(4)_R$ R-indices $a, b, \ldots = 1, \ldots, 4$. In the superfield formalism all these states are combined into a $\mathcal{N}=4$ superfield

$$\Phi^{\mathcal{N}=4} = g_+ + \eta_a f_+^a + \frac{1}{2!}\eta_a\eta_b s^{ab} + \frac{1}{3!}\eta_a\eta_b\eta_c f_-^{abc} + \eta_1\eta_2\eta_3\eta_4 g_-^{1234} \,, \tag{2.75}$$

where the η_a's are Grassmann variables also labeled by $SU(4)_R$ R-symmetry indices. The supercharges are represented by

$$\widetilde{q}_a = |p\rangle \eta_a \,, \qquad q^a = |p] \frac{\partial}{\partial \eta_a} \,, \tag{2.76}$$

and relate all the 16 states into one supermultiplet. For instance,

$$\begin{aligned} \widetilde{q}_1 \Phi^{\mathcal{N}=4} &= |p\rangle \left(\eta_1 g_+ + \eta_d \eta_a f_+^a + \frac{1}{2!}\eta_1\eta_a\eta_b s^{ab} + \frac{1}{3!}\eta_1\eta_a\eta_b\eta_c f_-^{abc} + \eta_1\eta_1\eta_2\eta_3\eta_4 g_-^{1234} \right) \\ &= \eta_1 |p\rangle g_+ + \eta_1\eta_a |p\rangle f_+^a + \frac{1}{2!}\eta_1\eta_a\eta_b |p\rangle s^{ab} + \frac{1}{3!}\eta_1\eta_a\eta_b\eta_c |p\rangle f_-^{abc} \,. \end{aligned} \tag{2.77}$$

The action of $\widetilde{q}_1$ on the individual states can be read off from this expression by comparing with the η-expansion in $\Phi^{\mathcal{N}=4}$. For example

$$g_+ \xrightarrow{\widetilde{q}_1} 0 \,, \qquad f_+^a \xrightarrow{\widetilde{q}_1} |p\rangle \delta_1^a g_+ \,, \qquad s^{ab} \xrightarrow{\widetilde{q}_1} |p\rangle 2!(\delta_1^a f_+^b - \delta_1^b f_+^a) \,, \qquad etc. \,, \tag{2.78}$$

gives the well-known supersymmetric transformation rules, see *e.g.* [50].

We can think of the Φ's as super-states and introduce a superamplitude

$$\mathcal{A}_n^{\mathcal{N}=4}(\Phi_1, \Phi_2, \ldots, \Phi_n) \,, \tag{2.79}$$

which represents a sum of amplitudes with all different helicity assignments and combinations of external states from the supermultiplet. The expansion coefficients, which uniquely identify a given component helicity amplitude, are precisely the η_a's, one set for each external line. Because the amplitudes must be invariant under $SU(4)_R$ R-symmetry, this puts constraints on the combination of indices that can occur for non-vanishing amplitudes. Many of the amplitudes in the direct expansion will therefore vanish, since they are $SU(4)_R$ R-symmetry violating. Schematically, what is left is an expansion of the form

$$\begin{aligned}\mathcal{A}_n^{\mathcal{N}=4} &= \mathcal{A}_n^{MHV,\mathcal{N}=4} + \mathcal{A}_n^{NMHV,\mathcal{N}=4} + \ldots + \mathcal{A}_n^{\overline{MHV},\mathcal{N}=4} \\ &= \sum A_n^{MHV}(\eta)^8 + \sum A_n^{NMHV}(\eta)^{12} + \ldots + \sum A_n^{\overline{MHV}}(\eta)^{4n-8}\,,\end{aligned} \tag{2.80}$$

where each $SU(4)_R$ R-symmetry index ($a = 1, 2, 3, 4$) appears the same number of times in each of the η-monomials. Here $\mathcal{A}_n^{N^kMHV,\mathcal{N}=4}$ denotes the superamplitude for the N^kMHV helicity sector and $A_n^{N^kMHV}$ are the actual component helicity amplitudes. The amplitudes with zero or four η variables vanish because they are directly related to vanishing amplitudes through super-Ward identities [37, 38]. The component amplitudes can be extracted from the superamplitudes by acting with the corresponding (Grassmann) differential operators (or integrals) to single out the desired components. As an example let us consider the MHV part, which can be written as, see *e.g.* [51],

$$\mathcal{A}_n^{MHV,\mathcal{N}=4} = \frac{\delta^{(8)}\left(\sum_{i=1}^n |i\rangle\eta_a\right)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,, \tag{2.81}$$

where

$$\delta^{(8)}\left(\sum_{i=1}^n |i\rangle\eta_a\right) = \frac{1}{2^4}\prod_{a=1}^4\sum_{i,j=1}^n \langle ij\rangle\eta_{ia}\eta_{ja}\,. \tag{2.82}$$

Assume we want to extract the purely gluonic MHV amplitude $A_n(g_1^-, g_2^-, g_3^+, \ldots, g_n^+)$. From the superfield in eq. (2.75) we can read off that this is the coefficient to the monomial

$$(\eta_{11}\eta_{12}\eta_{13}\eta_{14})(\eta_{21}\eta_{22}\eta_{23}\eta_{24})\,, \tag{2.83}$$

in eq. (2.81). Extracting this monomial from eq. (2.82) we find the Parke-Taylor formula

$$A_n(g_1^-, g_2^-, g_3^+, \ldots, g_n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,, \tag{2.84}$$

introduced in eq. (2.56). Likewise, if we wish to obtain the fermionic MHV amplitude $A_n(f_1^+, f_2^-, g_3^-, g_4^+, \ldots, g_n^+)$, we can extract it from the monomial

$$(\eta_{11})(\eta_{22}\eta_{23}\eta_{24})(\eta_{31}\eta_{32}\eta_{33}\eta_{34})\,, \tag{2.85}$$

and find

$$A_n(f_1^+, f_2^-, g_3^-, g_4^+, \ldots, g_n^+) = \frac{\langle 13\rangle\langle 23\rangle^3}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,. \tag{2.86}$$

The superamplitudes $\mathcal{A}_n^{\mathcal{N}=4}$ do also satisfy the same relations as those discussed in section 2.1 [18]. For instance, the photon-decoupling identity reads

$$\sum_{\sigma\in cyclic} \mathcal{A}_n^{\mathcal{N}=4}(1, \sigma(2, 3, \ldots, n)) = 0\,. \tag{2.87}$$

The corresponding identity for component amplitudes is obtained by collecting terms with the same η-monomial. However, one need to be careful with the sign in this process. Consider, for example, the η-monomials corresponding to the following four-point fermion amplitudes

$$\begin{aligned} A_4(g_1^-, f_2^+, f_3^-, g_4^+) &\sim (\eta_{11}\eta_{12}\eta_{13}\eta_{14})(\eta_{21})(\eta_{32}\eta_{33}\eta_{34})\,, \\ A_4(g_1^-, f_3^-, g_4^+, f_2^+) &\sim (\eta_{11}\eta_{12}\eta_{13}\eta_{14})(\eta_{32}\eta_{33}\eta_{34})(\eta_{21})\,, \\ A_4(g_1^-, g_4^+, f_2^+, f_3^-) &\sim (\eta_{11}\eta_{12}\eta_{13}\eta_{14})(\eta_{21})(\eta_{32}\eta_{33}\eta_{34})\,. \end{aligned} \tag{2.88}$$

Since the Grassmann numbers are anticommuting we have

$$(\eta_{11}\eta_{12}\eta_{13}\eta_{14})(\eta_{32}\eta_{33}\eta_{34})(\eta_{21}) = -(\eta_{11}\eta_{12}\eta_{13}\eta_{14})(\eta_{21})(\eta_{32}\eta_{33}\eta_{34})\,. \tag{2.89}$$

This means that the photon-decoupling identity between these amplitudes is

$$A_4(g_1^-, f_2^+, f_3^-, g_4^+) - A_4(g_1^-, f_3^-, g_4^+, f_2^+) + A_4(g_1^-, g_4^+, f_2^+, f_3^-) = 0\,, \tag{2.90}$$

i.e. with a sign change in the term where the order of the fermions has changed.

The $\mathcal{N}_G = 8$ supergravity theory has an on-shell formalism analogous to $\mathcal{N} = 4$ super-Yang-Mills theory. The supermultiplet of $\mathcal{N}_G = 8$ supergravity contains 2 gravitons $h_\pm$, 16 gravitinos $\psi_\pm$, 56 graviphotons $v_\pm$, 112 graviphotinos $\chi_\pm$ and 70 real scalars ϕ. These 256 states can be encoded into the following superfield

$$\begin{aligned} \Phi^{\mathcal{N}_G=8} = h_+ &+ \eta_A\psi_+^A + \frac{1}{2!}\eta_A\eta_B v_+^{AB} + \frac{1}{3!}\eta_A\eta_B\eta_C\chi_+^{ABC} + \frac{1}{4!}\eta_A\eta_B\eta_C\eta_D\phi^{ABCD} \\ &+ \frac{1}{5!}\eta_A\eta_B\eta_C\eta_D\eta_E\chi_-^{ABCDE} + \frac{1}{6!}\eta_A\eta_B\eta_C\eta_D\eta_E\eta_F v_-^{ABCDEF} \\ &+ \frac{1}{7!}\eta_A\eta_B\eta_C\eta_D\eta_E\eta_F\eta_G\psi_-^{ABCDEFG} + \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\eta_8 h_-^{12345678}\,, \end{aligned} \tag{2.91}$$

where $A, B, \ldots = 1, \ldots, 8$ are $SU(8)_R$ R-symmetry indices. The gravity superamplitude is denoted as

$$\mathcal{M}_n^{\mathcal{N}_G=8}(\Phi_1, \Phi_2, \ldots, \Phi_n)\,, \tag{2.92}$$

which, when expanded out in terms of the η_A's, gives a sum of all possible component amplitudes $M_n^{N^kMHV}$ dressed with the corresponding strings of η_A's. The $SU(8)_R$ invariance again dictates that only amplitudes dressed with a string of η_A's where each A index appears an equal amount of times can be non-vanishing. This fact will be important later on.

The Φ-Ψ Formalism

The superfield formalism can also be used for *less* than maximally supersymmetric theories. Contrary to the $\mathcal{N} = 4$ super-Yang-Mills and the $\mathcal{N}_G = 8$ supergravity cases the full CPT-invariant spectrum of particle states is not given by just *one* supermultiplet. Therefore it can not be collected into just one superfield either. For the minimal $\mathcal{N} < 4$ and $\mathcal{N}_G < 8$ theories we need two superfields. As introduced in [52, 53] these can be obtained from the maximally supersymmetric superfields by truncating or integrating out η variables.

For minimal $\mathcal{N} < 4$ super-Yang-Mills theory the two superfields can be obtained from

$$\begin{aligned} \Phi^{\mathcal{N}<4} &= \Phi^{\mathcal{N}=4}|_{\eta_{\mathcal{N}+1},\ldots,\eta_4\to 0}\,, \\ \Psi^{\mathcal{N}<4} &= \int \prod_{a=\mathcal{N}+1}^{4} d\eta_a \Phi^{\mathcal{N}=4}\,. \end{aligned} \tag{2.93}$$

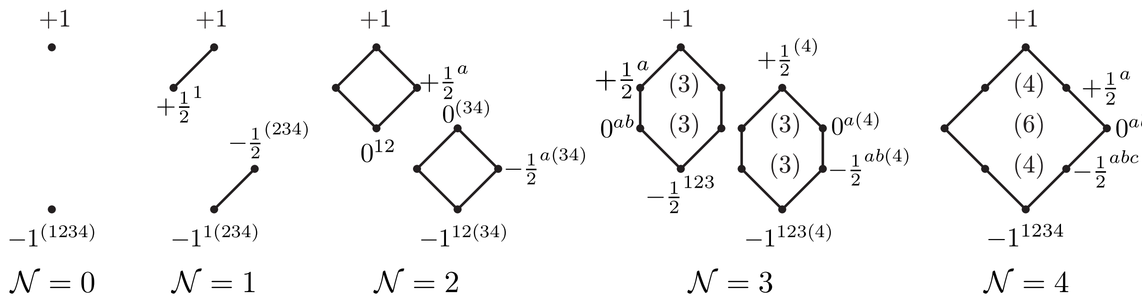


Figure 2.2: Diamond diagrams for the superfields of $\mathcal{N}=0,1,2,3,4$ super-Yang-Mills theories. The $SU(\mathcal{N})_R$ indices a,b,c are labeled as superscripts, where $a<b<c$ with $a,b,c=1,2,\ldots,\mathcal{N}$. The hidden indices are indicated in parentheses and the numbers inside the diamonds show the number of corresponding states on each horizontal line. Each diamond makes up the states that are directly related by supersymmetry transformations.

For example, for $\mathcal{N}=3$ we find

$$\begin{aligned}\Phi^{\mathcal{N}=3} &= g_+ + \eta_a f_+^a + \frac{1}{2!}\eta_a\eta_b s^{ab} + \eta_1\eta_2\eta_3 f_-^{123}\,,\\ \Psi^{\mathcal{N}=3} &= f_+^{(4)} - \eta_a s^{a(4)} + \frac{1}{2!}\eta_a\eta_b f_-^{ab(4)} - \eta_1\eta_2\eta_3 g_-^{123(4)}\,,\end{aligned} \tag{2.94}$$

where $a,b=1,2,3$, and for $\mathcal{N}=2$

$$\begin{aligned}\Phi^{\mathcal{N}=2} &= g_+ + \eta_a f_+^a + \eta_1\eta_2 s^{12}\,,\\ \Psi^{\mathcal{N}=2} &= -\,s^{(34)} - \eta_a f_-^{a(34)} - \eta_1\eta_2 g_-^{12(34)}\,,\end{aligned} \tag{2.95}$$

with $a=1,2$.

We write the indices corresponding to η variables that have been integrated out in parentheses. Although they have nothing to do with the $SU(\mathcal{N}<4)_R$ symmetry, we keep them for reasons that will be clear in chapter 6. Using the combined Φ-Ψ superfields, all states now have CPT-conjugate partners.

Note that the $\mathcal{N}=3$ case is a bit "artificial". It is well-known that the $\mathcal{N}=3$ and $\mathcal{N}=4$ super-Yang-Mills theories are identical, and looking at the combined particle content of the $\Phi^{\mathcal{N}=3}$ and $\Psi^{\mathcal{N}=3}$ fields in eq. (2.94), we see that it is indeed equal to the particle content of the $\mathcal{N}=4$ theory. We can even combine the two fields to give us back $\Phi^{\mathcal{N}=4}$

$$\Phi^{\mathcal{N}=4} = \Phi^{\mathcal{N}=3} + \eta_4\Psi^{\mathcal{N}=3}\,, \tag{2.96}$$

illustrating that the superfields of $\mathcal{N}=3$ are nothing but a rewriting of the $\mathcal{N}=4$ superfield.

Instead of writing down the explicit expressions for the superfields, like in eq. (2.94) and (2.95), we can represent them by "diamond diagrams" [28]. The superfields for $\mathcal{N}=0,1,2,3,4$ super-Yang-Mills theory are then represented by the diagrams shown in figure 2.2. This gives a more transparent overview of the particle content.

The truncation/integration can also be applied directly on superamplitudes, and thereby provide us with the $\mathcal{N}<4$ superamplitudes. Suppose the $i_1,i_2,\ldots,i_m$ external legs are in the Ψ superfield representation, while the remaining external legs are in the Φ representation. The $\mathcal{N} < 4$ superamplitude is then obtained from the $\mathcal{N} = 4$ superamplitude

$$\mathcal{A}^{\mathcal{N}<4}_{n,i_1\ldots i_m} = \left[\int \prod_{a_1=\mathcal{N}+1}^{4} d\eta_{i_1,a_1} \cdots \prod_{a_m=\mathcal{N}+1}^{4} d\eta_{i_m,a_m} \mathcal{A}^{\mathcal{N}=4}_n(\Phi_1, \Phi_2, \ldots, \Phi_n)\right]_{\eta_{\mathcal{N}+1},\ldots,\eta_4\to 0} . \tag{2.97}$$

This picks out the subset of amplitudes in eq. (2.80) belonging to the $\mathcal{N} < 4$ theory under consideration.

In the $\mathcal{N}_G < 8$ supergravity case we get the two superfields from

$$\begin{aligned} \Phi^{\mathcal{N}_G<8} &= \Phi^{\mathcal{N}_G=8}|_{\eta_{\mathcal{N}_G+1},\ldots,\eta_8\to 0} , \\ \Psi^{\mathcal{N}_G<8} &= \int \prod_{A=\mathcal{N}_G+1}^{8} d\eta_A \Phi^{\mathcal{N}_G=8} . \end{aligned} \tag{2.98}$$

This gives the following superfields for $\mathcal{N}_G = 6$ supergravity

$$\begin{aligned} \Phi^{\mathcal{N}_G=6} = h_+ &+ \sum_{i=1,2,3,4,5,6} \eta_i \psi^i_+ + \sum_{i<j=1,2,3,4,5,6} \eta_i\eta_j v^{ij}_+ + \sum_{i<j<k=1,2,3,4,5,6} \eta_i\eta_j\eta_k \chi^{ijk}_+ \\ &+ \sum_{i<j<k<l=1,2,3,4,5,6} \eta_i\eta_j\eta_k\eta_l \phi^{ijkl} + \sum_{i<j<k<l<m=1,2,3,4,5,6} \eta_i\eta_j\eta_k\eta_l\eta_m \chi^{ijklm}_- \\ &+ \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6 v^{123456}_- , \end{aligned} \tag{2.99}$$

representing the $(+2, +^3\!/\!_2{}^6, +1^{15}, +^1\!/\!_2{}^{20}, 0^{15}, -^1\!/\!_2{}^6, -1)$ multiplet, and

$$\begin{aligned} \Psi^{\mathcal{N}_G=6} = &- v^{(78)}_+ - \sum_{i=1,2,3,4,5,6} \eta_i \chi^{i(78)}_+ - \sum_{i<j=1,2,3,4,5,6} \eta_i\eta_j \phi^{ij(78)} \\ &- \sum_{i<j<k=1,2,3,4,5,6} \eta_i\eta_j\eta_k \chi^{ijk(78)}_- - \sum_{i<j<k<l=1,2,3,4,5,6} \eta_i\eta_j\eta_k\eta_l v^{ijkl(78)}_- \\ &- \sum_{i<j<k<l<m=1,2,3,4,5,6} \eta_i\eta_j\eta_k\eta_l\eta_m \psi^{ijklm(78)}_- - \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6 h^{123456(78)}_- , \end{aligned} \tag{2.100}$$

representing the $(+1, +^1\!/\!_2{}^6, 0^{15}, -^1\!/\!_2{}^{20}, -1^{15}, -^3\!/\!_2{}^6, -2)$ multiplet. The superscripts denote the degeneracy of states of the corresponding helicity.

Similar to the equivalence between $\mathcal{N} = 3$ and $\mathcal{N} = 4$ super-Yang-Mills theory, the superfields for $\mathcal{N}_G = 7$ supergravity are just a rewriting of the $\mathcal{N}_G = 8$ superfield, mirroring the fact that these two theories are identical.

We get the superamplitudes for $\mathcal{N}_G < 8$ supergravity by performing the truncations directly on the $\mathcal{N}_G = 8$ superamplitude

$$\mathcal{M}^{\mathcal{N}_G<8}_{n,i_1\ldots i_m} = \left[\int \prod_{A_1=\mathcal{N}_G+1}^{8} d\eta_{i_1,A_1} \cdots \prod_{A_m=\mathcal{N}_G+1}^{8} d\eta_{i_m,A_m} \mathcal{M}^{\mathcal{N}_G=8}_n(\Phi_1, \Phi_2, \ldots, \Phi_n)\right]_{\eta_{\mathcal{N}_G+1},\ldots,\eta_8\to 0} , \tag{2.101}$$

where again the $i_1, i_2, \ldots, i_m$ external legs are in the Ψ superfield representation.

In chapter 6 we will examine the effect of other kinds of truncations than those introduced above, and their implications for the linear symmetry groups.

Chapter 3

Monodromy and BCJ-Relations

In this chapter we present a new class of tree-level amplitude relations. Contrary to those reviewed in chapter 2, these do not directly follow from the color group. Although in this thesis we are primarily concerned with field theories, these relations have such a beautiful derivation in terms of open-string amplitudes, that we first present the explicit derivations from this point of view [15, 16]. The corresponding field theory relations can then readily be obtained by taking the $\alpha' \to 0$ limit. Afterward we will see how these can also be derived recursively through the BCFW recursion relation. Both this and the next chapter therefore illustrate how string theory can teach us something about field theory structures.

3.1 Monodromy Relations

String theory is a mathematical framework in which fundamental particles are described as one-dimensional oscillating objects, so-called strings. As a string propagates through space-time it sweeps out a two-dimensional surface known as a world sheet. Scattering processes correspond to the joining and splitting of strings which can either be open or closed. In this chapter we will only be concerned with tree-level scattering of *open* strings, whose world sheet can be mapped to the upper half of the complex plane H_+.

External open-string states give rise to a spectrum of gauge bosons. They are represented by vertex operators inserted at positions x_i on the boundary of the world sheet, *i.e.* on the real axis when mapped to H_+. The amplitude is essentially obtained by integrating a correlation function of the vertex operators over all possible insertions x_i. Considering color-ordered amplitudes the ordering is dictated by the integration region. It is by deformation of these world-sheet integrations we will derive the monodromy relations among subamplitudes of open strings.

After evaluating the correlation functions of the vertex operators the color-ordered open-string amplitudes can be written as [16, 19, 54]

$$\mathcal{A}_n(1,\ldots,n) = \int \prod_{i=1}^{n} dx_i \, \frac{|w_{ab}\, w_{ac}\, w_{bc}|}{dx_a dx_b dx_c} \prod_{i=1}^{n-1} H(x_{i+1}-x_i) \prod_{1\leq i<j\leq n} |x_i - x_j|^{\alpha' k_i \cdot k_j} \, F_n \,, \tag{3.1}$$

with

$$w_{ij} = x_i - x_j \,. \tag{3.2}$$

The ordering of the external legs is enforced by the product of Heaviside functions such that

$$H(x) = \begin{cases} 0 & x < 0 \,, \\ 1 & x \geq 0 \,. \end{cases} \tag{3.3}$$

The $SL(2,\mathbb{R})$ invariance requires that we fix the position of three points, here denoted x_a, x_b and x_c. A traditional choice is $x_1 = 0$, $x_{n-1} = 1$ and $x_n = +\infty$, and the amplitude is then

$$\mathcal{A}_n(1,\ldots,n) = \int \prod_{i=2}^{n-2} dx_i |x_i|^{\alpha' k_1\cdot k_i} |x_i - 1|^{\alpha' k_{n-1}\cdot k_i} \prod_{i<j\leq n-2} |x_j - x_i|^{\alpha' k_i\cdot k_j} H(x_{i+1}-x_i)\, F_n\,. \tag{3.4}$$

The helicity dependence and type of external states are contained in the F_n factor. For tachyons $F_n = 1$, while for gauge bosons, with polarization vectors ε_i, one has

$$F_n = \exp\left(-\sum_{i\neq j}\left(\frac{\sqrt{\alpha'}(\varepsilon_i\cdot k_j)}{(x_i-x_j)} - 2\frac{(\varepsilon_i\cdot\varepsilon_j)}{(x_i-x_j)^2}\right)\right)\Bigg|_{\text{multilinear in } \varepsilon_i}\,. \tag{3.5}$$

Note that the above F_n for gauge bosons does not depend on the ordering of external legs and when expanded out the additional $(x_i - x_j)$ factors will have powers of $-n_{ij}$ with n_{ij} being an integer. These factors will therefore not have any effect on the branch cuts and thereby on the analysis below. Everything that follows is valid for general amplitudes, but for ease of notation we just write down the tachyonic version in this section.

Let us begin the derivation of the monodromy relations with some explicit lower-point examples.

Four-Point Relations

The four-point open-string amplitudes are very simple, they involve only one integral,

$$\begin{aligned}
\mathcal{A}_4(2,1,3,4) &= \int_{-\infty}^{0} dx_2 (-x_2)^{\alpha' k_1\cdot k_2}(1-x_2)^{\alpha' k_2\cdot k_3}\,,\\
\mathcal{A}_4(1,2,3,4) &= \int_{0}^{1} dx_2 (x_2)^{\alpha' k_1\cdot k_2}(1-x_2)^{\alpha' k_2\cdot k_3}\,,\\
\mathcal{A}_4(1,3,2,4) &= \int_{1}^{\infty} dx_2 (x_2)^{\alpha' k_1\cdot k_2}(x_2-1)^{\alpha' k_2\cdot k_3}\,.
\end{aligned} \tag{3.6}$$

If we make an analytic continuation of the x_2 variable to the complex plane we can consider integrating the integrand of $\mathcal{A}_4(2,1,3,4)$ over the closed contour shown in figure 3.1. Since no poles lie inside the contour the residue theorem tells us that this integral is zero. Taking the contour to infinity the upper semicircle will not contribute and we have

$$\begin{aligned}
0 = &\int_{-\infty}^{0} dx_2(-x_2)^{\alpha' k_1\cdot k_2}(1-x_2)^{\alpha' k_2\cdot k_3} + \int_0^1 dx_2(-x_2)^{\alpha' k_1\cdot k_2}(1-x_2)^{\alpha' k_2\cdot k_3}\\
&+ \int_1^{\infty} dx_2(-x_2)^{\alpha' k_1\cdot k_2}(1-x_2)^{\alpha' k_2\cdot k_3}\,.
\end{aligned} \tag{3.7}$$

Comparing with the expressions in (3.6) this can be rewritten in terms of the open-string amplitudes as

$$\begin{aligned}
\mathcal{A}_4(2,1,3,4) = &-\int_0^1 dx_2(-x_2)^{\alpha' k_1\cdot k_2}(1-x_2)^{\alpha' k_2\cdot k_3} - \int_1^{\infty} dx_2(-x_2)^{\alpha' k_1\cdot k_2}(1-x_2)^{\alpha' k_2\cdot k_3}\\
= &- e^{i\pi\alpha' k_1\cdot k_2}\int_0^1 dx_2 (x_2)^{\alpha' k_1\cdot k_2}(1-x_2)^{\alpha' k_2\cdot k_3}\\
&- e^{i\pi\alpha'(k_1\cdot k_2 + k_2\cdot k_3)}\int_1^{\infty} dx_2 (x_2)^{\alpha' k_1\cdot k_2}(x_2-1)^{\alpha' k_2\cdot k_3}\\
= &- e^{i\pi\alpha' k_1\cdot k_2}\mathcal{A}_4(1,2,3,4) - e^{i\pi\alpha'(k_1\cdot k_2+k_2\cdot k_3)}\mathcal{A}_4(1,3,2,4)\,.
\end{aligned} \tag{3.8}$$

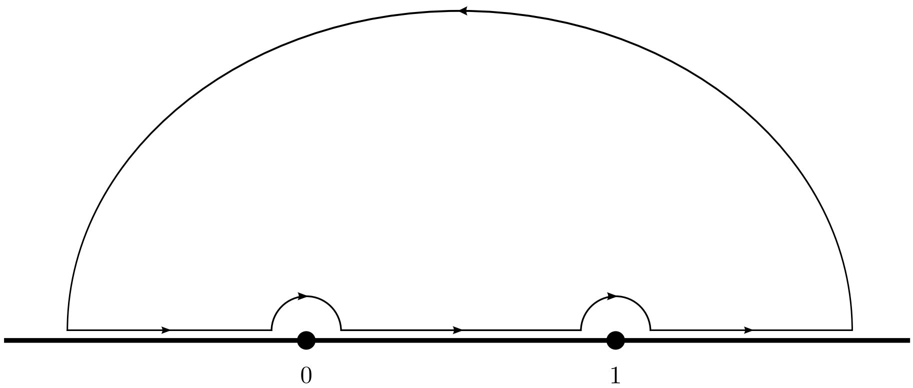


Figure 3.1: Contour of x_2-integration over the integrand of $\mathcal{A}_4(2,1,3,4)$.

We have pulled out exponential factors, in correspondence with the branch cuts, every time we pass one of the branch points. This is done in order write the arguments of the power functions inside the integrand, such that they can be directly related to the amplitudes in (3.6). For more details on branch points and cuts see appendix A. Eq. (3.8) is what we call a monodromy relation between four-point open-string amplitudes. The real part of this relation is

$$\mathcal{A}_4(2,1,3,4) = -\cos(\pi\alpha' k_1 \cdot k_2)\mathcal{A}_4(1,2,3,4) - \cos(\pi\alpha' k_2 \cdot (k_1 + k_3))\mathcal{A}_4(1,3,2,4)\,, \quad (3.9)$$

which is the string version of the photon-decoupling relation. Indeed, letting $\alpha' \to 0$ the cosines go to 1 and the amplitudes to their corresponding field-theory expressions, and we are left with

$$A_4(2,1,3,4) = -A_4(1,2,3,4) - A_4(1,3,2,4)\,. \quad (3.10)$$

The imaginary part of eq. (3.8) is

$$0 = -\sin(\pi\alpha' k_1 \cdot k_2)\mathcal{A}_4(1,2,3,4) - \sin(\pi\alpha' k_2 \cdot (k_1 + k_3))\mathcal{A}_4(1,3,2,4)\,, \quad (3.11)$$

which can be rewritten as

$$\mathcal{A}_4(1,3,2,4) = \frac{\sin(\pi\alpha' k_1 \cdot k_2)}{\sin(\pi\alpha' k_2 \cdot k_4)}\mathcal{A}_4(1,2,3,4)\,. \quad (3.12)$$

In the $\alpha' \to 0$ limit we get

$$A_4(1,3,2,4) = \frac{k_1 \cdot k_2}{k_2 \cdot k_4} A_4(1,2,3,4) = \frac{s_{12}}{s_{24}} A_4(1,2,3,4)\,. \quad (3.13)$$

This is a new relation among color-ordered amplitudes not included in any of those we reviewed in section 2.1. In particular we see that this relation reduces the number of independent color-ordered four-point amplitudes to one, whereas the KK-relations only implied a reduction to two. Let us see how this generalizes to five points.

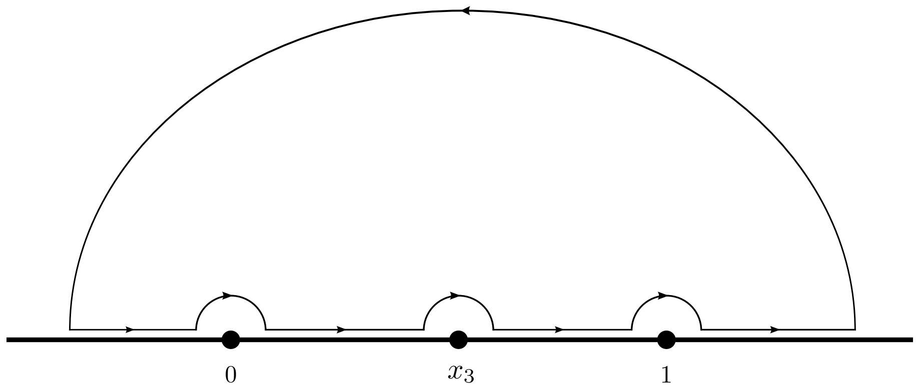


Figure 3.2: Contour of x_2-integration over $I(x_2)$.

Five-Point Relations

Consider the five-point amplitude

$$\begin{aligned}\mathcal{A}_5(2,1,3,4,5) = &\int_{-\infty}^{0} dx_2 \int_0^1 dx_3 (-x_2)^{\alpha' k_1\cdot k_2}(x_3)^{\alpha' k_1\cdot k_3} \\ &\qquad\qquad \times (1-x_2)^{\alpha' k_4\cdot k_2}(1-x_3)^{\alpha' k_4\cdot k_3}(x_3-x_2)^{\alpha' k_2\cdot k_3} \\ \equiv &\int_{-\infty}^{0} dx_2 I(x_2)\,. \end{aligned} \tag{3.14}$$

Again we analytically continue the x_2-integration of $I(x_2)$ to the complex plane, and look at an integration over the contour shown in figure 3.2. Like for the four-point case this gives us a relation

$$\mathcal{A}_5(2,1,3,4,5) = -\int_0^{x_3} dx_2 I(x_2) - \int_{x_3}^{1} dx_2 I(x_2) - \int_1^{\infty} dx_2 I(x_2)\,, \tag{3.15}$$

with

$$\int_0^{x_3} dx_2 I(x_2) = e^{i\pi\alpha' k_1\cdot k_2}\mathcal{A}_5(1,2,3,4,5)\,, \tag{3.16}$$

$$\int_{x_3}^{1} dx_2 I(x_2) = e^{i\pi\alpha' k_2\cdot(k_1+k_3)}\mathcal{A}_5(1,3,2,4,5)\,, \tag{3.17}$$

$$\int_1^{\infty} dx_2 I(x_2) = e^{i\pi\alpha' k_2\cdot(k_1+k_3+k_4)}\mathcal{A}_5(1,3,4,2,5)\,, \tag{3.18}$$

and hence

$$\begin{aligned}\mathcal{A}_5(2,1,3,4,5) = &-e^{i\pi\alpha' k_1\cdot k_2}\mathcal{A}_5(1,2,3,4,5) - e^{i\pi\alpha' k_2\cdot(k_1+k_3)}\mathcal{A}_5(1,3,2,4,5) \\ &- e^{i\pi\alpha' k_2\cdot(k_1+k_3+k_4)}\mathcal{A}_5(1,3,4,2,5)\,. \end{aligned} \tag{3.19}$$

The real part of eq. (3.19) is

$$\begin{aligned}\mathcal{A}_5(2,1,3,4,5) = &-\cos(\pi\alpha' k_1\cdot k_2)\mathcal{A}_5(1,2,3,4,5) - \cos(\pi\alpha' k_2\cdot(k_1+k_3))\mathcal{A}_5(1,3,2,4,5) \\ &-\cos(\pi\alpha' k_2\cdot(k_1+k_3+k_4))\mathcal{A}_5(1,3,4,2,5)\,, \end{aligned} \tag{3.20}$$

which in the $\alpha' \to 0$ limit is again the photon-decoupling identity

$$A_5(2,1,3,4,5) = -A_5(1,2,3,4,5) - A_5(1,3,2,4,5) - A_5(1,3,4,2,5)\,. \tag{3.21}$$

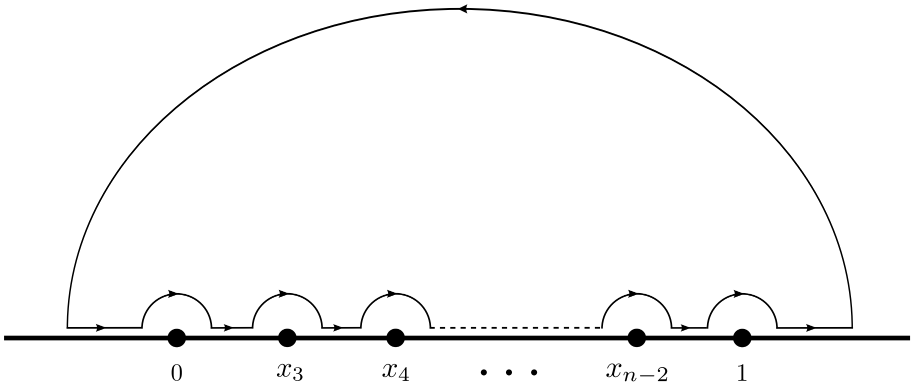


Figure 3.3: Contour of x_2-integration.

The imaginary part is

$$
\begin{aligned}
0 = \;& \sin(\pi\alpha' k_1 \cdot k_2)\mathcal{A}_5(1,2,3,4,5) + \sin(\pi\alpha' k_2 \cdot (k_1 + k_3))\mathcal{A}_5(1,3,2,4,5) \\
& + \sin(\pi\alpha' k_2 \cdot (k_1 + k_3 + k_4))\mathcal{A}_5(1,3,4,2,5)\,,
\end{aligned} \tag{3.22}
$$

and in the field-theory limit it becomes

$$
0 = s_{12}A_5(1,2,3,4,5) + (s_{12} + s_{23})A_5(1,3,2,4,5) + (s_{12} + s_{23} + s_{24})A_5(1,3,4,2,5)\,. \tag{3.23}
$$

This relation is again of a different nature than the ones we discussed earlier. With these, and the relations obtained by permutation of labels, the number of independent color-ordered five-point amplitudes reduces to two, compared to the six left after applying only KK-relations. From these examples we can see how to generalize this to n points.

N-Point Relations

Take the n-point amplitude

$$
\mathcal{A}_n(2,1,3,4,\ldots,n) = \int \prod_{i=2}^{n-2} dx_i |x_i|^{\alpha' k_1 \cdot k_i} |x_i - 1|^{\alpha' k_{n-1} \cdot k_i} \prod_{i<j\leq n-2} |x_j - x_i|^{\alpha' k_i \cdot k_j}\,, \tag{3.24}
$$

where the integration is over $x_2 < 0 < x_3 < \cdots < x_{n-2} < 1$, and change the integration contour for x_2 as shown in figure 3.3. Every time x_2 passes a point x_j it picks up an additional phase factor $e^{i\pi\alpha' k_2 \cdot k_j}$ when relating it to color-ordered amplitudes. This leads to the following relation

$$
\begin{aligned}
0 = \;& \mathcal{A}_n(2,1,3,\ldots,n) + e^{i\pi\alpha' k_2\cdot k_1}\mathcal{A}_n(1,2,3,\ldots,n) + e^{i\pi\alpha' k_2\cdot(k_1+k_3)}\mathcal{A}_n(1,3,2,4,\ldots,n) \\
& + \cdots + e^{i\pi\alpha' k_2\cdot(k_1+k_3+\cdots+k_{n-1})}\mathcal{A}_n(1,3,4,\ldots,n-1,2,n)\,.
\end{aligned} \tag{3.25}
$$

Like in the previous cases, the $\alpha' \to 0$ limit takes the real part to the photon-decoupling identity

$$
\begin{aligned}
0 = \;& A_n(2,1,3,\ldots,n) + A_n(1,2,3,\ldots,n) + A_n(1,3,2,4,\ldots,n) \\
& + \cdots + A_n(1,3,4,\ldots,n-1,2,n)\,,
\end{aligned} \tag{3.26}
$$

while the imaginary part gives rise to a new kind of relation

$$\begin{aligned} 0 = s_{12}A_n(1,2,3,\ldots,n) + (s_{12}+s_{23})A_n(1,3,2,4,\ldots,n) \\ + \cdots + (s_{12}+s_{23}+\cdots+s_{2(n-1)})A_n(1,3,4,\ldots,n-1,2,n)\,, \end{aligned} \tag{3.27}$$

reducing the number of independent color-ordered n-point amplitudes from $(n-2)!$ down to $(n-3)!$.

In the field-theory limit, relations like eq. (3.27) are known as Bern-Carrasco-Johansson (BCJ) relations [12]. They were the first to conjecture this kind of identities (although from a completely different point of view). Only later were the above string-theory derivation discovered. The BCJ-relations can also be proven without resorting to string theory. Their validity at lower points can easily be checked, and, through the BCFW recursion relation, the higher-point relations can then be shown recursively [17]. We will illustrate this in the next section. Here we will just note that in the same way as the gauge-theory relations in section 2.1 could be extended to $\mathcal{N}=4$ super-Yang-Mills theory, so can the BCJ-relations [14, 18]. In terms of superamplitudes they read

$$\begin{aligned} 0 = s_{12}\mathcal{A}_n^{\mathcal{N}=4}(1,2,3,\ldots,n) + (s_{12}+s_{23})\mathcal{A}_n^{\mathcal{N}=4}(1,3,2,4,\ldots,n) \\ + \cdots + (s_{12}+s_{23}+\cdots+s_{2(n-1)})\mathcal{A}_n^{\mathcal{N}=4}(1,3,4,\ldots,n-1,2,n)\,, \end{aligned} \tag{3.28}$$

from where all BCJ-relations among component helicity amplitudes can be extracted. Like in the photon-decoupling identities, care must be taken when the relation involves fermions changing position.

BCJ-Relations from BCFW Recursion

Since the expressions looks rather messy when written down for general n points, we will, for simplicity, only prove the four- and five-point versions of eq. (3.27) explicitly. From this the general procedure for n points should be clear and will be schematically outlined. This proof is a slightly modified version of the one that can be found in [17].

Let us start with the four-point case, *i.e.* we wish to prove

$$0 = s_{12}A_4(1,2,3,4) + (s_{12}+s_{23})A_4(1,3,2,4)\,. \tag{3.29}$$

Make a BCFW-shift in 1 and 4 and consider the following contour integral

$$I_4 \equiv -\oint_\infty \frac{dz}{z} s_{\widehat{1}2}A_4(\widehat{1},3,\widehat{4},2)\,. \tag{3.30}$$

Since $\widehat{1},\widehat{4}$ are not adjacent the amplitude behaves as $1/z^2$ for $z\to\infty$, see *e.g.* [55], and the integral is therefore zero, *i.e.* $I_4=0$, even with the extra $s_{\widehat{1}2}$ factor in the integrand. Using the photon-decoupling identity it can also be written as

$$I_4 = \oint_\infty \frac{dz}{z} s_{\widehat{1}2}(A_4(\widehat{1},2,3,\widehat{4}) + A_4(\widehat{1},3,2,\widehat{4}))\,. \tag{3.31}$$

Introducing the short-hand notation

$$A_4(\widehat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,\widehat{4}) \equiv \sum_{h=\pm} A_3(\widehat{1},2,-\widehat{P}_{12}^h)\frac{1}{P_{12}^2}A_3(\widehat{P}_{12}^{-h},3,\widehat{4})\,, \tag{3.32}$$

etc., it follows from the residue theorem that

$$\begin{aligned}I_4 &= s_{12}(A_4(1,2,3,4) + A_4(1,3,2,4)) \\ &\quad - s_{\widehat{1}2}(z_{12})A_4(\widehat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,\widehat{4}) - s_{\widehat{1}2}(z_{13})A_4(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,\widehat{4}) \\ &= s_{12}A_4(1,2,3,4) + (s_{12} - s_{\widehat{1}2}(z_{13}))A_4(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,\widehat{4})\,, \end{aligned} \tag{3.33}$$

where we have used BCFW to write $A_4(1,3,2,4) = A_4(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,\widehat{4})$. We have also put $s_{\widehat{1}2}(z_{12})A_4(\widehat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,\widehat{4}) = 0$, because $s_{\widehat{1}2}(z_{12}) = 0$, but for the generalization to higher points it might be better to think of this as a consequence of the three-point "BCJ-relation" $s_{\widehat{1}2}(z_{12})A_3(\widehat{1},2,-\widehat{P}_{12}) = 0$. Likewise we are allowed to add the term $-s_{2\widehat{4}}(z_{13})A_4(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,\widehat{4})$, because of the "BCJ-relation" $s_{2\widehat{4}}(z_{13})A_3(\widehat{P}_{13},2,\widehat{4}) = 0$, to get

$$\begin{aligned}I_4 &= s_{12}A_4(1,2,3,4) + (s_{12} - [s_{\widehat{1}2}(z_{13}) + s_{2\widehat{4}}(z_{13})])A_4(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,\widehat{4}) \\ &= s_{12}A_4(1,2,3,4) + (s_{12} + s_{23})A_4(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,\widehat{4})\,. \end{aligned} \tag{3.34}$$

We used momentum conservation to write $s_{\widehat{1}2}(z_{13}) + s_{2\widehat{4}}(z_{13}) = -s_{23}$. Since $A_4(1,3,2,4) = A_4(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,\widehat{4})$, and we know that $I_4 = 0$, this is exactly the four-point BCJ-relation

$$0 = s_{12}A_4(1,2,3,4) + (s_{12} + s_{23})A_4(1,3,2,4)\,. \tag{3.35}$$

Let us see how this generalizes to the five-point BCJ-relation

$$0 = s_{12}A_5(1,2,3,4,5) + (s_{12}+s_{23})A_5(1,3,2,4,5) + (s_{12}+s_{23}+s_{24})A_5(1,3,4,2,5)\,. \tag{3.36}$$

Make a BCFW-shift in leg 1 and 5 and consider the contour integral

$$I_5 \equiv -\oint_\infty \frac{dz}{z} s_{\widehat{1}2}A_5(\widehat{1},3,4,\widehat{5},2) = 0\,, \tag{3.37}$$

which is again zero due to the shifted legs being non-adjacent. Using the photon-decoupling identity, and afterward the residue theorem, we can write it as

$$\begin{aligned}I_5 &= \oint_\infty \frac{dz}{z} s_{\widehat{1}2}(A_5(\widehat{1},2,3,4,\widehat{5}) + A_5(\widehat{1},3,2,4,\widehat{5}) + A_5(\widehat{1},3,4,2,\widehat{5})) \\ &= s_{12}(A_5(1,2,3,4,5) + A_5(1,3,2,4,5) + A_5(1,3,4,2,5)) \\ &\quad - s_{\widehat{1}2}(z_{12})A_5(\widehat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,4,\widehat{5}) - s_{\widehat{1}2}(z_{123})A_5(\widehat{1},2,3,-\widehat{P}_{123}|\widehat{P}_{123},4,\widehat{5}) \\ &\quad - s_{\widehat{1}2}(z_{13})A_5(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,4,\widehat{5}) - s_{\widehat{1}2}(z_{132})A_5(\widehat{1},3,2,-\widehat{P}_{132}|\widehat{P}_{132},4,\widehat{5}) \\ &\quad - s_{\widehat{1}2}(z_{13})A_5(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},4,2,\widehat{5}) - s_{\widehat{1}2}(z_{134})A_5(\widehat{1},3,4,-\widehat{P}_{134}|\widehat{P}_{134},2,\widehat{5})\,. \end{aligned} \tag{3.38}$$

By BCFW we can write

$$A_5(1,2,3,4,5) = A_5(\widehat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,4,\widehat{5}) + A_5(\widehat{1},2,3,-\widehat{P}_{123}|\widehat{P}_{123},4,\widehat{5})\,, \tag{3.39}$$

and likewise for $A_5(1,3,2,4,5)$ and $A_5(1,3,4,2,5)$. Let us define $I_5 \equiv I_5^3 + I_5^4$, where I_5^3 is all the terms with a three-point amplitude to the left in the above splitting, and I_5^4 all terms with a four-point amplitude to the left, *i.e.*

$$\begin{aligned}I_5^3 &\equiv s_{12}\big[A_5(\widehat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,4,\widehat{5}) + A_5(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,4,\widehat{5}) + A_5(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},4,2,\widehat{5})\big] \\ &\quad - s_{\widehat{1}2}(z_{12})A_5(\widehat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,4,\widehat{5}) - s_{\widehat{1}2}(z_{13})A_5(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,4,\widehat{5}) \\ &\quad - s_{\widehat{1}2}(z_{13})A_5(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},4,2,\widehat{5}) \\ &= (s_{12} - s_{\widehat{1}2}(z_{12}))A_5(\widehat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,4,\widehat{5}) + (s_{12} - s_{\widehat{1}2}(z_{13}))A_5(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,4,\widehat{5}) \\ &\quad + (s_{12} - s_{\widehat{1}2}(z_{13}))A_5(\widehat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},4,2,\widehat{5})\,. \end{aligned} \tag{3.40}$$

In the first term we again have $s_{\hat{1}2}(z_{12})=0$, and because of the four-point BCJ-relation $s_{2\hat{5}}(z_{13})A_4(\widehat{P}_{13},4,2,\hat{5})+(s_{2\hat{5}}(z_{13})+s_{24})A_4(\widehat{P}_{13},2,4,\hat{5})=0$, we can add

$$-s_{2\hat{5}}(z_{13})A_5(\hat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},4,2,\hat{5})-(s_{2\hat{5}}(z_{13})+s_{24})A_5(\hat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,4,\hat{5})\,, \tag{3.41}$$

to eq. (3.40)

$$\begin{aligned}I_5^3&=s_{12}A_5(\hat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,4,\hat{5})\\&\quad+(s_{12}-[s_{\hat{1}2}(z_{13})+s_{2\hat{5}}(z_{13})+s_{24}])A_5(\hat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,4,\hat{5})\\&\quad+(s_{12}-[s_{\hat{1}2}(z_{13})+s_{2\hat{5}}(z_{13})])A_5(\hat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},4,2,\hat{5})\,.\end{aligned} \tag{3.42}$$

It follows from momentum conservation that $s_{\hat{1}2}(z_{13})+s_{2\hat{5}}(z_{13})+s_{24}=-s_{23}$ and $s_{\hat{1}2}(z_{13})+s_{2\hat{5}}(z_{13})=-s_{23}-s_{24}$, hence

$$\begin{aligned}I_5^3&=s_{12}A_5(\hat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,4,\hat{5})\\&\quad+(s_{12}+s_{23})A_5(\hat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,4,\hat{5})\\&\quad+(s_{12}+s_{23}+s_{24})A_5(\hat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},4,2,\hat{5})\,.\end{aligned} \tag{3.43}$$

For I_5^4 we have

$$\begin{aligned}I_5^4&=(s_{12}-s_{\hat{1}2}(z_{123}))A_5(\hat{1},2,3,-\widehat{P}_{123}|\widehat{P}_{123},4,\hat{5})\\&\quad+(s_{12}-s_{\hat{1}2}(z_{132}))A_5(\hat{1},3,2,-\widehat{P}_{132}|\widehat{P}_{132},4,\hat{5})\\&\quad+(s_{12}-s_{\hat{1}2}(z_{134}))A_5(\hat{1},3,4,-\widehat{P}_{134}|\widehat{P}_{134},2,\hat{5})\,.\end{aligned} \tag{3.44}$$

Using a four-point BCJ-relation we can add

$$s_{\hat{1}2}(z_{123})A_5(\hat{1},2,3,-\widehat{P}_{123}|\widehat{P}_{123},4,\hat{5})+(s_{\hat{1}2}(z_{123})+s_{23})A_5(\hat{1},3,2,-\widehat{P}_{132}|\widehat{P}_{132},4,\hat{5})=0\,, \tag{3.45}$$

to eq. (3.44), and get

$$\begin{aligned}I_5^4&=s_{12}A_5(\hat{1},2,3,-\widehat{P}_{123}|\widehat{P}_{123},4,\hat{5})+(s_{12}+s_{23})A_5(\hat{1},3,2,-\widehat{P}_{132}|\widehat{P}_{132},4,\hat{5})\\&\quad+(s_{12}-s_{\hat{1}2}(z_{134}))A_5(\hat{1},3,4,-\widehat{P}_{134}|\widehat{P}_{134},2,\hat{5})\,.\end{aligned} \tag{3.46}$$

In the last term we use momentum conservation to write $-s_{\hat{1}2}(z_{134})=s_{23}+s_{24}+s_{2\hat{5}}(z_{134})=s_{23}+s_{24}+s_{\widehat{134}}(z_{134})=s_{23}+s_{24}$, *i.e.*

$$\begin{aligned}I_5^4&=s_{12}A_5(\hat{1},2,3,-\widehat{P}_{123}|\widehat{P}_{123},4,\hat{5})\\&\quad+(s_{12}+s_{23})A_5(\hat{1},3,2,-\widehat{P}_{132}|\widehat{P}_{132},4,\hat{5})\\&\quad+(s_{12}+s_{23}+s_{24})A_5(\hat{1},3,4,-\widehat{P}_{134}|\widehat{P}_{134},2,\hat{5})\,.\end{aligned} \tag{3.47}$$

This finally give us

$$\begin{aligned}I_5&=I_5^3+I_5^4\\&=s_{12}\big[A_5(\hat{1},2,-\widehat{P}_{12}|\widehat{P}_{12},3,4,\hat{5})+A_5(\hat{1},2,3,-\widehat{P}_{123}|\widehat{P}_{123},4,\hat{5})\big]\\&\quad+(s_{12}+s_{23})\big[A_5(\hat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},2,4,\hat{5})+A_5(\hat{1},3,2,-\widehat{P}_{132}|\widehat{P}_{132},4,\hat{5})\big]\\&\quad+(s_{12}+s_{23}+s_{24})\big[A_5(\hat{1},3,-\widehat{P}_{13}|\widehat{P}_{13},4,2,\hat{5})+A_5(\hat{1},3,4,-\widehat{P}_{134}|\widehat{P}_{134},2,\hat{5})\big]\,,\end{aligned} \tag{3.48}$$

where each of the $[\cdots]$ terms are a BCFW expansion of the corresponding color-ordered amplitudes, *i.e.*

$$0 = I_5 = s_{12}A_5(1,2,3,4,5) + (s_{12}+s_{23})A_5(1,3,2,4,5) + (s_{12}+s_{23}+s_{24})A_5(1,3,4,2,5)\,, \tag{3.49}$$

proving the five-point BCJ-relation.

The steps for general n points should be clear now. Make a BCFW-shift in 1 and n and start from the contour integral

$$I_n \equiv -\oint_\infty \frac{dz}{z} s_{\widehat{1}2} A_n(\widehat{1},3,4,\ldots,\widehat{n},2) = 0\,. \tag{3.50}$$

Using the photon-decoupling identity I_n can be rewritten as

$$I_n = \oint_\infty \frac{dz}{z} s_{\widehat{1}2}[A_n(\widehat{1},2,3,4,\ldots,\widehat{n}) + A_n(\widehat{1},3,2,4,\ldots,\widehat{n}) + \ldots + A_n(\widehat{1},3,4,\ldots,2,\widehat{n})]\,. \tag{3.51}$$

From the residue theorem this can be split into

$$I_n = \sum_{i=3}^{n-1} I_n^i\,, \tag{3.52}$$

where I_n^i represents all terms with an i-point amplitude to the left in the amplitude factorization. Using lower-point BCJ-relations, and momentum conservation, each I_n^i can be written in the form of eq. (3.27) with the residues of the amplitudes (instead of the full color-ordered amplitudes) corresponding to the i under consideration. The sum of all I_n^i then gives the full BCFW-expansion of each of the amplitudes dressed with the appropriate kinematic factor $(s_{12}+s_{23}+\ldots s_{2j})$. This is exactly the n-point BCJ-relation.

The supersymmetric BCJ-relations in eq. (3.28) can be proven in a similar way [18], using the supersymmetric extension of the BCFW recursion relations.

Minimal Basis

Although eq. (3.25) has a very nice n-point form, it does not make the reduction to only $(n-3)!$ independent amplitudes obvious. Above we focused on one specific way of changing contours, but we could equally well consider other deformations which would lead to different versions of the monodromy relations. We will now consider the same type of deformation as used in [15] to show the $(n-3)!$ reduction.

Consider a n-point amplitude $\mathcal{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n)$ with r points $\{\beta_1,\ldots,\beta_r\}$ in the interval $]-\infty,0[$, $k-1$ points $\{\alpha_1,\ldots,\alpha_{k-1}\}$ in $]0,1[$, and $s-k$ points $\{\alpha_{k+1},\ldots,\alpha_s\}$ in $]1,\infty[$. For ease of notation we have defined $x_{\alpha_k} \equiv x_{n-1} = 1$. Here $r+(k-1)+(s-k) = r-1+s = n-3$. Note that there is no loss of generality in assuming leg $n-1$ to sit in what we have called the α set, since this ordering can always be obtained by use of cyclicity and reflection symmetry (and redefinition of what we denote as the α and β sets).

In the same way as above we can analytically continue the contours in the $]-\infty,0[$ region, *i.e.* those related to legs in the β set, and write them in terms of integrals in the $]0,\infty[$ region. The sum of integrals in the $]0,\infty[$ region can in turn be related to amplitudes of different ordering multiplied by the appropriate phase factor. One arrives at the following result [15]

$$\mathcal{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n) = (-1)^r \prod_{1\le i<j\le r} e^{i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\in\mathrm{OP}(\{\alpha\}\cup\{\beta^T\})} \prod_{i=0}^{s}\prod_{j=1}^{r} e^{(\alpha_i,\beta_i)}\mathcal{A}_n(1,\sigma,n)\,, \tag{3.53}$$

with $e^{(\alpha,\beta)} \equiv e^{i\pi\alpha' k_\alpha\cdot k_\beta}$ if $x_\beta > x_\alpha$ and 1 otherwise, α_0 denotes leg 1 (at point 0).

Let us write out a couple of examples of this formula explicitly. At four points with $\{\alpha\}=\{3\}$ and $\{\beta\}=\{2\}$ one gets

$$\mathcal{A}_4(2,1,3,4) = -e^{i\pi\alpha' k_2\cdot(k_1+k_3)}\mathcal{A}_4(1,3,2,4) - e^{i\pi\alpha' k_1\cdot k_2}\mathcal{A}_4(1,2,3,4)\,. \tag{3.54}$$

The real part of this equation expresses $\mathcal{A}_4(2,1,3,4)$ in terms of the two amplitudes $\mathcal{A}_4(1,\sigma(2,3),4)$, while the imaginary part relates these two to each other, thus reducing the number of independent color-ordered amplitudes to one.

At five points we get two types of relations; one with $\{\beta\}=\{2,3\}$ and $\{\alpha\}=\{4\}$

$$\begin{aligned}\mathcal{A}_5(2,3,1,4,5) = e^{i\pi\alpha' k_2\cdot k_3} e^{i\pi\alpha' k_1\cdot(k_2+k_3)} \Big[& e^{i\pi\alpha' k_4\cdot(k_2+k_3)}\mathcal{A}_5(1,4,3,2,5) \\ & + e^{i\pi\alpha' k_4\cdot k_2}\mathcal{A}_5(1,3,4,2,5) + \mathcal{A}_5(1,3,2,4,5)\Big]\,, \end{aligned} \tag{3.55}$$

and the other with $\{\beta\}=\{2\}$ and $\{\alpha\}=\{3,4\}$

$$\begin{aligned}\mathcal{A}_5(2,1,3,4,5) = -e^{i\pi\alpha' k_1\cdot k_2} \Big[& e^{i\pi\alpha' k_2\cdot(k_3+k_4)}\mathcal{A}_5(1,3,4,2,5) \\ & + e^{i\pi\alpha' k_2\cdot k_3}\mathcal{A}_5(1,3,2,4,5) + \mathcal{A}_5(1,2,3,4,5)\Big]\,, \end{aligned} \tag{3.56}$$

along with the two obtained by interchanging $2 \leftrightarrow 3$.

Using the real part of these equations the basis of independent subamplitudes have been reduced to the six amplitudes $\mathcal{A}_5(1,\sigma(2,3,4),5)$. By means of the imaginary part one can use eq. (3.56) to relate $\mathcal{A}_5(1,\sigma(2),4,\sigma(3),5)$ to $\mathcal{A}_5(1,\sigma(2,3),4,5)$, and then use this in eq. (3.55) to also express $\mathcal{A}_5(1,4,\sigma(2,3),5)$ solely in terms of $\mathcal{A}_5(1,\sigma(2,3),4,5)$, thereby reducing to a basis of merely two subamplitudes.

The general argument goes as follows; first one uses the real part of eq. (3.53) to reduce to the $(n-2)!$ basis of amplitudes given by $\mathcal{A}_n(1,\sigma,n)$. Next one can use the imaginary part of eq. (3.53), *i.e*

$$0 = \mathrm{Im}\left[\prod_{1\le i<j\le r} e^{i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\in\mathrm{OP}(\{\alpha\}\cup\{\beta^T\})} \prod_{i=0}^{s}\prod_{j=1}^{r} e^{(\alpha_i,\beta_i)}\mathcal{A}_n(1,\sigma,n)\right]\,, \tag{3.57}$$

to relate amplitudes of the form $\mathcal{A}_n(1,\sigma(2,\ldots,n-3),n-1,\sigma(n-2),n)$ to $\mathcal{A}_n(1,\sigma(2,\ldots,n-2),n-1,n)$, through the relations with only one leg in the β set. This can then be used in the relations with two legs in the β set to express amplitudes of the form $\mathcal{A}_n(1,\sigma(2,\ldots,n-4),n-1,\sigma(n-3,n-2),n)$ in terms of $\mathcal{A}_n(1,\sigma(2,\ldots,n-2),n-1,n)$ *etc.*. Thereby recursively relate all amplitudes to the $(n-3)!$ basis of $\mathcal{A}_n(1,\sigma(2,\ldots,n-2),n-1,n)$.

In the field-theory limit the real part of eq. (3.53) is

$$A_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n) = (-1)^r \sum_{\sigma\in\mathrm{OP}(\{\alpha\}\cup\{\beta^T\})} A_n(1,\sigma,n)\,, \tag{3.58}$$

which is just the KK-relations (compared to eq. (2.18) after a cyclic permutation on the left-hand side).

Before taking the field-theory limit of the imaginary part we can rewrite it a bit

$$\begin{aligned}
&\prod_{1\leq i<j\leq r} e^{i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\in\mathrm{OP}(\{\alpha\}\cup\{\beta^T\})} \prod_{i=0}^{s}\prod_{j=1}^{r} e^{(\alpha_i,\beta_i)} \mathcal{A}_n(1,\sigma,n) \\
&= \sum_{\sigma\in\mathrm{OP}(\{\alpha\}\cup\{\beta^T\})} \exp\left(\frac{i\pi\alpha'}{2}\sum_{1\leq i<j\leq r} s_{\beta_i\beta_j} + \sum_{i=0}^{s}\sum_{j=1}^{r} i\pi\alpha'(\alpha_i,\beta_i)\right) \mathcal{A}_n(1,\sigma,n) \\
&= \sum_{\sigma\in\mathrm{OP}(\{\alpha\}\cup\{\beta^T\})} \exp\left(\frac{i\pi\alpha'}{2}\sum_{i=1}^{r}\sum_{\sigma_J<\sigma_{\beta_i}} s_{\beta_i J}\right) \mathcal{A}_n(1,\sigma,n)\,.
\end{aligned} \tag{3.59}$$

Here σ_J denotes the position of leg J in the permutation σ, and we define $\sigma_1 = 0$. We can then write eq. 3.57 in the field-theory limit as

$$0 = \sum_{\sigma\in\mathrm{OP}(\{\alpha\}\cup\{\beta^T\})} \sum_{i=1}^{r}\sum_{\sigma_J<\sigma_{\beta_i}} s_{\beta_i J} A_n(1,\sigma,n)\,. \tag{3.60}$$

The same argument as used above to reduce to a basis of $(n-3)!$ subamplitudes still works in the $\alpha' \to 0$ limit. A BCFW proof for eq. (3.60) was given in [56], where the original conjectured BCJ-relations are proven as well.

3.2 The BCJ-Representation

In last section we saw how one can derive relations among string-theory amplitudes by deforming contours of integration. From these, new gauge-theory relations were obtained in the field-theory limit. Despite the simplicity of the above string derivation, the relations were first conjectured by Bern, Carrasco and Johansson in field theory [12]. Their basis for this came from a somewhat surprising duality between color and kinematic factors. In this section we will present this *color-kinematic duality* and illustrate how it connects to the BCJ-relations.

Motivation

The approach taken by BCJ can be motivated from some observations at four points. We start by forcing the color-ordered amplitudes into a form corresponding to having only antisymmetric three-point vertices

$$A_4(1,2,3,4) = \frac{n_s}{s} + \frac{n_t}{t}\,, \quad A_4(1,2,4,3) = -\frac{n_u}{u} - \frac{n_s}{s}\,, \quad A_4(1,3,2,4) = -\frac{n_t}{t} + \frac{n_u}{u}\,. \tag{3.61}$$

Here $s \equiv s_{12}$, $t \equiv s_{14}$, $u \equiv s_{13}$, and any four-point contact terms have been absorbed into the numerators, using trivial relations like $s/s = t/t = u/u = 1$. The relative signs have been chosen in accordance with the antisymmetry of the cubic vertices. This is quite straightforward to do at four points. Also note that one has the freedom to shift the numerators

$$n_s \longrightarrow n_s + \rho s\,, \qquad n_t \longrightarrow n_t - \rho t\,, \qquad n_u \longrightarrow n_u - \rho u\,, \tag{3.62}$$

without changing the amplitudes. Here ρ is just some arbitrary function.

If we plug this into eq. (2.13), for $n = 4$, we find

$$\begin{aligned}\mathcal{A}_4 = (ig)^2 &\left[\underbrace{\widetilde{f}^{a_1a_2x_1}\widetilde{f}^{x_1a_3a_4}}_{c_s} A_4(1,2,3,4) + \underbrace{\widetilde{f}^{a_1a_3x_1}\widetilde{f}^{x_1a_2a_4}}_{c_u} A_4(1,3,2,4)\right] \\ =&(ig)^2 \left[c_s\left(\frac{n_s}{s} + \frac{n_t}{t}\right) + c_u\left(-\frac{n_t}{t} + \frac{n_u}{u}\right)\right] \\ =&(ig)^2 \left[\frac{c_s n_s}{s} + \frac{(c_s - c_u)n_t}{t} + \frac{c_u n_u}{u}\right] ,\end{aligned} \tag{3.63}$$

where, using the Jacobi identity, we can write

$$\underbrace{\widetilde{f}^{a_1a_2x_1}\widetilde{f}^{x_1a_3a_4}}_{c_s} - \underbrace{\widetilde{f}^{a_1a_3x_1}\widetilde{f}^{x_1a_2a_4}}_{c_u} = \widetilde{f}^{a_2a_3x_1}\widetilde{f}^{x_1a_4a_1} \equiv c_t\,, \tag{3.64}$$

i.e.

$$\mathcal{A}_4 = (ig)^2 \left[\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}\right] . \tag{3.65}$$

With the amplitude written in this form, not only do the color factors satisfy the Jacobi identity, but surprisingly so will the kinematic numerators. That is, we have a kind of duality

$$c_s - c_u = c_t \qquad \longleftrightarrow \qquad n_s - n_u = n_t\,, \tag{3.66}$$

between the color and the kinematic factors, see figure 3.4. At four points the Jacobi identity for the kinematic numerators is basically automatic if one forces the subamplitudes into the form of (3.61). In particular we see that the transformation in eq. (3.62) does not change the validity of the Jacobi identity

$$n_s - n_u - n_t = 0 \qquad \longrightarrow \qquad n_s - n_u - n_t + \rho\,\underbrace{(s+t+u)}_{0} = 0\,. \tag{3.67}$$

That the kinematic Jacobi identity is related to the four-point BCJ-relation from the last section, is easily seen by

$$\begin{aligned}0 &= s_{12}A_4(1,2,3,4) + (s_{12} + s_{23})A_4(1,3,2,4) \\ &= s\left(\frac{n_s}{s} + \frac{n_t}{t}\right) - u\left(-\frac{n_t}{t} + \frac{n_u}{u}\right) \\ &= n_s - n_u + n_t\frac{s+u}{t} \\ &= n_s - n_u - n_t\,,\end{aligned} \tag{3.68}$$

where we used $s + t + u = 0$.

At Higher Points

One might believe that the above duality between color factors and kinematic numerators is merely a curious coincidence at four points. After all the kinematics are very simple in that case. However, it turns out that this is a much more general structure. To be more

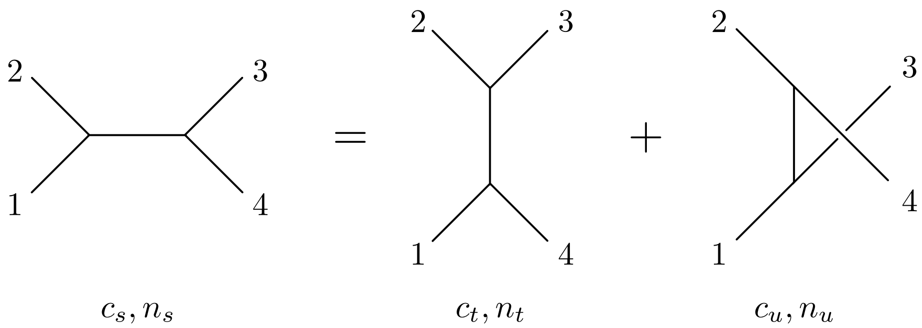


Figure 3.4: Diagrammatic representation of the Jacobi identity. The c_i's can be obtained by dressing the vertices with structure constants $\tilde{f}^{abc}$, but the diagrams can also be thought of as representing the kinematic numerators n_i.

explicit, we can represent the color-ordered five-point amplitudes in terms of diagrams with only antisymmetric three-point vertices, *i.e.*

$$A_5(1,2,3,4,5) = \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}, \tag{3.69}$$

$$A_5(1,4,3,2,5) = \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}}, \tag{3.70}$$

$$A_5(1,3,4,2,5) = \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}}, \tag{3.71}$$

$$A_5(1,2,4,3,5) = \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}}, \tag{3.72}$$

$$A_5(1,4,2,3,5) = \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}}, \tag{3.73}$$

$$A_5(1,3,2,4,5) = \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}}. \tag{3.74}$$

Here we have written down only the amplitudes with the position of leg 1 and 5 fixed. This is an independent set under the KK-relations in eq. (2.18). However, we emphasize that such a representation is completely consistent with the relations we reviewed in section 2.1. For example, consider the $A_5(1,2,3,5,4)$ amplitude, which can be written in terms of the amplitudes (3.69), (3.72) and (3.73) through the KK-relation

$$\begin{aligned} A_5(1,2,3,5,4) =& -A_5(1,2,3,4,5) - A_5(1,2,4,3,5) - A_5(1,4,2,3,5) \\ =& -\left(\frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}\right) \\ & -\left(\frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}}\right) \\ & -\left(\frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}}\right) \\ =& -\frac{n_1}{s_{12}s_{45}} - \frac{n_4}{s_{45}s_{23}} - \frac{n_{12}}{s_{12}s_{35}} - \frac{n_{14}}{s_{14}s_{35}} + \frac{n_7}{s_{23}s_{14}}. \end{aligned} \tag{3.75}$$

The last line of eq. (3.75) is the same expression as we find when writing $A_5(1,2,3,5,4)$ in terms of its five different diagrams only involving antisymmetric three-point vertices. In this sense the KK-relations can actually be inferred from such a representation. We point this out here, since this idea will reappear in chapter 7 in a slightly different setting.

Using eq. (3.69)–(3.74) in eq. (2.13), for $n = 5$, gives

$$\mathcal{A}_5 = (ig)^3 \left(\frac{c_1 n_1}{s_{12}s_{45}} + \frac{c_2 n_2}{s_{23}s_{51}} + \frac{c_3 n_3}{s_{34}s_{12}} + \frac{c_4 n_4}{s_{45}s_{23}} + \frac{c_5 n_5}{s_{51}s_{34}} + \frac{c_6 n_6}{s_{14}s_{25}} \right.$$
$$+ \frac{c_7 n_7}{s_{32}s_{14}} + \frac{c_8 n_8}{s_{25}s_{43}} + \frac{c_9 n_9}{s_{13}s_{25}} + \frac{c_{10} n_{10}}{s_{42}s_{13}} + \frac{c_{11} n_{11}}{s_{51}s_{42}} + \frac{c_{12} n_{12}}{s_{12}s_{35}}$$
$$\left. + \frac{c_{13} n_{13}}{s_{35}s_{24}} + \frac{c_{14} n_{14}}{s_{14}s_{35}} + \frac{c_{15} n_{15}}{s_{13}s_{45}} \right), \tag{3.76}$$

where the color factors, c_i, can be obtained from the 15 cubic-vertex-diagrams by dressing each vertex with a $\tilde{f}^{abc}$ factor. These factors are related through 9 (independent) Jacobi identities. Amazingly it is also possible to find solutions for the numerators, n_i, such that these obey the same Jacobi identities. So for each color-factor relation $c_\alpha + c_\beta - c_\gamma = 0$ we have a similar numerator relation $n_\alpha + n_\beta - n_\gamma = 0$.

That this Jacobi structure ensures the BCJ-relations again can be seen using eq. (3.69), (3.71) and (3.74) in

$$0 = s_{12}A_5(1,2,3,4,5) + (s_{12} + s_{23})A_5(1,3,2,4,5) + (s_{12} + s_{23} + s_{24})A_5(1,3,4,2,5)\,. \tag{3.77}$$

With a little rewriting this takes the form [57, 58]

$$0 = \frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_9 + n_{15}}{s_{13}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} - \frac{n_3 - n_5 + n_8}{s_{34}}\,. \tag{3.78}$$

Similar forms can be obtained for the other versions of the five-point BCJ-relations. We see that if the numerators satisfy the same Jacobi identities as the corresponding color factors, each of the four terms above is zero and the relation is satisfied. However, unlike in the four-point case, we also note that there is still freedom to have numerators which do not satisfy the Jacobi identities but still satisfy eq. (3.78). The important part is that solutions *can* be found where the color-kinematic duality is satisfied.

Bern, Carrasco and Johansson promoted this to a general principle saying; it is *always* possible to represent a n-point gauge-theory amplitude as a sum over all distinct n-point diagrams with only antisymmetric cubic vertices, in such a way that the kinematic numerators n_i satisfy the same Jacobi identities as the color factors c_i. In detail, this is to write the full tree-level amplitude in the form

$$\mathcal{A}_n(1,2,\ldots,n) = (ig)^{n-2} \sum_i \frac{c_i n_i}{(\prod_j s_j)_i}\,, \tag{3.79}$$

where the color-kinematic duality

$$c_\alpha + c_\beta - c_\gamma = 0 \qquad \longleftrightarrow \qquad n_\alpha + n_\beta - n_\gamma = 0\,, \tag{3.80}$$

is satisfied. The sum in eq. (3.79) is over all different diagrams, which only contain cubic vertices, and $(\prod_j s_j)_i$ is the corresponding pole structure of diagram i. Once again we stress that in the four point case the duality in (3.80) was satisfied automatically once the amplitude was cast into the form of eq. (3.65). This is *not* the case for general n-point amplitudes. To write an amplitude in the form of eq. (3.79) is not difficult, but to *also* have the numerators satisfy (3.80) is a highly non-trivial task, see *e.g.* [58–60].

Recently there have also been some interesting progress in understanding the duality at a more fundamental level, looking for an underlying kinematic group [61–63].

As illustrated above, the constraint of a Jacobi structure on the numerators dictates the existence of amplitude relations like eq. (3.77). This was what originally inspired BCJ to search for and find a n-point formula that expresses every subamplitude in terms of the $(n-3)!$ subamplitudes $A_n(1,2,3,\sigma(4,\ldots,n))$. The explicit expression for their formula is considerably more complicated than those considered earlier, but the content is the same.

In chapter 5 and 7 we will return to the BCJ-representation and see that there is still more structures and interesting generalizations to it.

Chapter 4

Factorization of Closed-String Amplitudes

In the last chapter we looked at relations involving only open-string (or gauge-theory) amplitudes. In this chapter we will turn our attention to relations between closed- and open-string tree-level amplitudes. Closed-string states contain a massless spin-2 particle which can be identified with the graviton. These relations thereby provide a connection between gravity and gauge theories. We will use the same methods as the original Kawai-Lewellen-Tye paper [19], factorizing closed-string amplitudes into products of open-string amplitudes, but write the final results in the form obtained in [64, 65].

The world sheet of a tree-level scattering process involving n closed-string states can be mapped to the complex plane, with the closed-string analog of eq. (3.4) being

$$\mathcal{M}_n = \left(\frac{i}{2\pi\alpha'}\right)^{n-3} \int \prod_{i=2}^{n-2} d^2 z_i |z_i|^{2\alpha' k_1\cdot k_i} |z_i - 1|^{2\alpha' k_{n-1}\cdot k_i} \prod_{i<j\leq n-2} |z_j - z_i|^{2\alpha' k_i\cdot k_j} F(z_i)\, G(\bar{z}_i)\,, \tag{4.1}$$

where we have fixed the three points $z_1 = 0$, $z_{n-1} = 1$ and $z_n = \infty$. Like for the open string, the $F(z_i)$ and $G(\bar{z}_i)$ functions depend on the type and helicity of the external states, however, since they are again without branch cuts they will not be important for the following discussion. We will denote $z_i = v_i^1 + i v_i^2$, such that

$$|z_i|^{2\alpha' k_1\cdot k_i} = \left[(v_i^1)^2 + (v_i^2)^2\right]^{\alpha' k_1\cdot k_i}\,, \tag{4.2}$$

$$|z_i - 1|^{2\alpha' k_{n-1}\cdot k_i} = \left[(v_i^1 - 1)^2 + (v_i^2)^2\right]^{\alpha' k_{n-1}\cdot k_i}\,, \tag{4.3}$$

$$|z_j - z_i|^{2\alpha' k_i\cdot k_j} = \left[(v_j^1 - v_i^1)^2 + (v_j^2 - v_i^2)^2\right]^{\alpha' k_i\cdot k_j}\,. \tag{4.4}$$

By making an analytic continuation of the v_i^2 variables to the complex plane, we can rotate the integration contour for these variables from the real axis to (almost) the imaginary axis

$$v_i^2 \quad\longrightarrow\quad i e^{-2i\epsilon} v_i^2 \simeq i(1 - 2i\epsilon) v_i^2\,, \tag{4.5}$$

without changing the value of the amplitude. Here $\epsilon > 0$ is some small number making sure we avoid the branch points. This changes the expressions in the integrand (to linear order in ϵ)

$$\left[(v_i^1)^2 + (v_i^2)^2\right]^{\alpha' k_1\cdot k_i} \longrightarrow \left[(v_i^1)^2 - (v_i^2)^2 + 4i\epsilon (v_i^2)^2\right]^{\alpha' k_1\cdot k_i}\,, \tag{4.6}$$

$$\left[(v_i^1 - 1)^2 + (v_i^2)^2\right]^{\alpha' k_{n-1}\cdot k_i} \longrightarrow \left[(v_i^1)^2 - (v_i^2)^2 - 2v_i^1 + 1 + 4i\epsilon (v_i^2)^2\right]^{\alpha' k_{n-1}\cdot k_i}\,, \tag{4.7}$$

$$\left[(v_j^1 - v_i^1)^2 + (v_j^2 - v_i^2)^2\right]^{\alpha' k_i\cdot k_j} \longrightarrow \left[(v_j^1 - v_i^1)^2 - (v_j^2 - v_i^2)^2 (1 - 4i\epsilon)\right]^{\alpha' k_i\cdot k_j}\,. \tag{4.8}$$

If we then make a transformation of variables

$$v_i^{\pm} \equiv v_i^1 \pm v_i^2 \,, \tag{4.9}$$

and define $\delta_i \equiv v_i^+ - v_i^-$, it is easy to verify that the expressions on the right-hand side of line (4.6), (4.7) and (4.8) are given by

$$(v_i^+ - i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}(v_i^- + i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}\,, \quad (v_i^+ - 1 - i\epsilon\delta_i)^{\alpha' k_{n-1}\cdot k_i}(v_i^- - 1 + i\epsilon\delta_i)^{\alpha' k_{n-1}\cdot k_i}\,, \tag{4.10}$$

and

$$\big(v_i^+ - v_j^+ - i\epsilon(\delta_i - \delta_j)\big)^{\alpha'\, k_i\cdot k_j}\big(v_i^- - v_j^- + i\epsilon(\delta_i - \delta_j)\big)^{\alpha'\, k_i\cdot k_j}\,, \tag{4.11}$$

respectively.

In total, this brings eq. (4.1) into the form

$$\begin{aligned}
\mathcal{M}_n = {} & \left(\frac{i}{2}\right)^{n-3}\left(\frac{i}{2\pi\alpha'}\right)^{n-3}\int_{-\infty}^{+\infty}\prod_{i=2}^{n-2} dv_i^+ dv_i^- F(v_i^-)\, G(v_i^+) \\
& \times (v_i^+ - i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}(v_i^- + i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}(v_i^+ - 1 - i\epsilon\delta_i)^{\alpha'\, k_{n-1}\cdot k_i}(v_i^- - 1 + i\epsilon\delta_i)^{\alpha'\, k_{n-1}\cdot k_i} \\
& \times \prod_{i<j\le n-2}\big(v_i^+ - v_j^+ - i\epsilon(\delta_i - \delta_j)\big)^{\alpha'\, k_i\cdot k_j}\big(v_i^- - v_j^- + i\epsilon(\delta_i - \delta_j)\big)^{\alpha'\, k_i\cdot k_j}\,,
\end{aligned} \tag{4.12}$$

where the additional factor of $(i/2)^{n-3}$ is due to the Jacobian when changing variables and from the rotation of the v_i^2 contours. The integrand of eq. (4.12) corresponds to a product of the integrands of two open-string amplitudes. The interesting result by Kawai, Lewellen and Tye was that a factorization also exists at the integrated level.

To obtain this, we first note the following; assume that at least one $v_i^+ \in]-\infty, 0[$ and look at the contribution from v_i^-, *i.e.*

$$\int_{-\infty}^{+\infty} dv_i^- F(v_i^-)(v_i^- + i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}(v_i^- - 1 + i\epsilon\delta_i)^{\alpha'\, k_{n-1}\cdot k_i}\prod_{i<j\le n-2}\big(v_i^- - v_j^- + i\epsilon(\delta_i - \delta_j)\big)^{\alpha'\, k_i\cdot k_j}\,. \tag{4.13}$$

The behaviour of the imaginary ϵ-terms near the branch points is

$$\begin{aligned}
v_i^- \sim 0 \quad &\Longrightarrow \quad \delta_i \sim v_i^+ < 0\,, \\
v_i^- \sim 1 \quad &\Longrightarrow \quad \delta_i \sim v_i^+ - 1 < 0\,, \\
v_i^- \sim v_j^- \quad &\Longrightarrow \quad \delta_i - \delta_j \sim v_i^+ - v_j^+ < 0 \quad \text{when} \quad v_i^+ < v_j^+.
\end{aligned} \tag{4.14}$$

The requirement $v_i^+ < v_j^+$ in the last line is of no concern, since we can choose to look at the v_i^- integral corresponding to the "smallest" v_i^+ variable, which has to lie in the range $]-\infty, 0[$ due to our first assumption. This behaviour implies that we can close the integral of v_i^- in the lower half of the complex v_i^--plane (again by analytical continuation), and since the closed contour does not contain any poles the integral vanishes. In general, when $v_i^+ < v_j^+$ we avoid the branch point $v_i^- = v_j^-$ below the real axis, and when $v_i^+ > v_j^+$ we avoid it above the real axis. From this kind of argument we see that whenever one of the v_i^+-variables is in the range of $]-\infty, 0[$ or $]1, \infty[$, at least one of the v_i^- contours can be completely closed either below or above the real axis. Hence, only when all v_i^+ lie between 0 and 1 will there be a contribution to eq. (4.12).

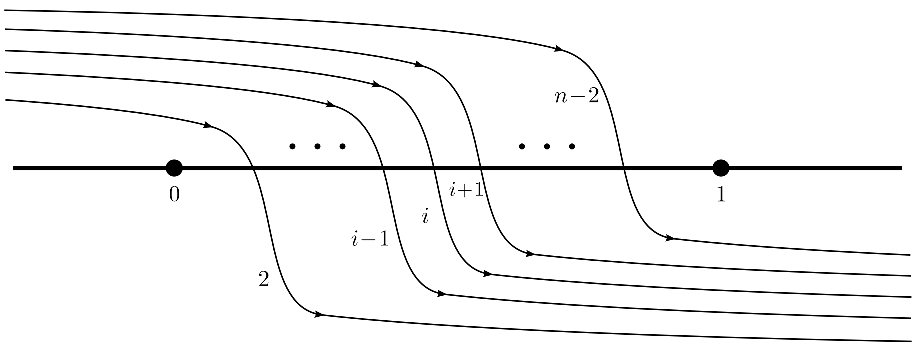


Figure 4.1: The nested structure of the contours of integration for the v_i^- variables corresponding to the ordering $0 < v_2^+ < v_3^+ < \cdots < v_{n-2}^+ < 1$ of the v_i^+ variables.

By splitting up the v_i^+-integration region we can write the n-point closed string amplitude as

$$\mathcal{M}_n = \sum_\sigma M_n^\sigma\big(\sigma(2), \ldots, \sigma(n-2)\big)\,, \tag{4.15}$$

where $M_n^\sigma(\sigma(2), \cdots, \sigma(n-2))$ is the "ordered amplitude" defined such that $v^+_{\sigma(2)} < v^+_{\sigma(3)} < \cdots < v^+_{\sigma(n-2)}$. For instance, at five points this corresponds to splitting the integration over the (v_2^+, v_3^+)-plane into an integral over the region "above" the $v_3^+ = v_2^+$ line (*i.e.* $v_2^+ < v_3^+$) and an integral "below" this line (*i.e.* $v_3^+ < v_2^+$). Together with the above restriction on the v_i^+-integration range, the v_i^+ part of M_n^σ in eq. (4.15) becomes

$$\begin{aligned} \int_{0<v^+_{\sigma(2)}<\cdots<v^+_{\sigma(n-2)}<1} \prod_{i=2}^{n-2} dv_i^+\, G\big(v_i^+\big)\big(v_i^+\big)^{\alpha' k_1\cdot k_i}\big(1-v_i^+\big)^{\alpha'\,k_{n-1}\cdot k_i} \\ \times \prod_{i<j\le n-2} \big(v^+_{\sigma(j)} - v^+_{\sigma(i)}\big)^{\alpha'\,k_{\sigma(i)}\cdot k_{\sigma(j)}}\,, \end{aligned} \tag{4.16}$$

where we have omitted writing the infinitesimal ϵ-terms explicitly. We recognize (4.16) as the expression corresponding to the color-ordered open-string amplitude $\mathcal{A}_n(1, \sigma(2, \ldots, n-2), n-1, n)$. Note that, compared to eq. (4.12), we have written $(1-v_i^+)^{\alpha'\,k_{n-1}\cdot k_i}$ instead of $(v_i^+ - 1)^{\alpha'\,k_{n-1}\cdot k_i}$ and $(v_j^+ - v_i^+)^{\alpha'\,k_i\cdot k_j}$ such that $v_j^+ - v_i^+ > 0$ always. This is needed in order to make the identification with a color-ordered open-string amplitude, however, we are only allowed to do this if we make a similar change in the v_i^- part, otherwise we get wrong phase factors.

For simplicity we will from now on fix the ordering to $\{2, 3, \ldots, n-2\}$, *i.e.* we are considering the $M_n^\sigma(2, 3, \ldots, n-2)$ contribution in eq. (4.15). The remaining terms can be obtained through permutation of labels.

We have just seen that the v_i^+ part is nothing but the color-ordered $\mathcal{A}_n(1, 2, \ldots, n)$ amplitude. We now turn to the v_i^- part, investigating which contours the imaginary ϵ terms dictate for the integrals. Near $v_i^- \sim 0$ the quantity $i\epsilon\delta_i \sim i\epsilon v_i^+$ is a positive imaginary number (remember that $v_i^+ \in]0, 1[$), so the contour is above the real axis here. For $v_i^- \sim 1$ we have $i\epsilon\delta_i \sim i\epsilon(v_i^+ - 1)$ which is a negative imaginary number, hence the contour lies below the real axis. Finally, for $v_i^- \sim v_j^-$ we see that $i\epsilon(\delta_i - \delta_j) \sim i\epsilon(v_i^+ - v_j^+)$, meaning that the contour for $v_i^- + i\epsilon\delta_i$ lies below the contour of $v_j^- + i\epsilon\delta_j$ for $i < j$. See figure 4.1 for an illustration of this nested structure.

The next step is to deform the contours for $v_i^- + i\epsilon\delta_i$ to form expressions corresponding to color-ordered amplitudes. That is, we are going to close the contours either to the left, turning the contour below the real axis, or to the right, turning the contour above the real axis. Besides having the correct integration *region*, in order to identify the integrals with open-string amplitudes, we also need to make sure that the *integrand* is correct. This implies that we sometimes need to change signs in the arguments of power functions, thereby pulling out phase factors similarly to what we did when deriving the monodromy relations. In order not to cross a branch cut we do this in the following way; for z^c with $\mathrm{Re}(z) < 0$ and with the branch cut lying on the negative real axis

$$z^c = \begin{cases} e^{i\pi c}(-z)^c & \mathrm{Im}(z) \geq 0\,, \\ e^{-i\pi c}\,(-z)^c & \mathrm{Im}(z) < 0\,. \end{cases} \tag{4.17}$$

When the branch cut lies on the positive real axis we instead have

$$z^c = \begin{cases} e^{-i\pi c}(-z)^c & \mathrm{Im}(z) \geq 0\,, \\ e^{i\pi c}\,(-z)^c & \mathrm{Im}(z) < 0\,. \end{cases} \tag{4.18}$$

For additional details see appendix A.

Furthermore, there is freedom as to how many contours we close to the left or the right. For a given $2 \leq j \leq n-1$, we can pull the contours from 2 up to $j-1$ to the left, and the set from j to $n-2$ to the right ($j = 2$ or $j = n-1$ means *all* to the right or *all* to the left, respectively). Before going to n points, let us illustrate this for the four- and five-point cases.

4.1 Four-Point KLT-Relations

In the four-point case eq. (4.16) is

$$\mathcal{A}_4(1,2,3,4) = \int_0^1 dv_2^+\, G\big(v_2^+\big)\big(v_2^+\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^+\big)^{\alpha'\, k_3\cdot k_2}\,, \tag{4.19}$$

and the v_2^- part is

$$\int_{-\infty}^{\infty} dv_2^-\, F\big(v_2^-\big)\big(v_2^-\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^-\big)^{\alpha'\, k_3\cdot k_2}\,. \tag{4.20}$$

Note that we write $(1-v_2^-)^{\alpha' k_3\cdot k_2}$ instead of $(v_2^- - 1)^{\alpha' k_3\cdot k_2}$ in order to compensate for the same swapping of order in the v_2^+ integral of eq. (4.16).

We start by considering the case where the v_2^- contour is pulled to the left, *i.e.* $j = 3$, see figure 4.2,

$$\begin{aligned}
&\int_{-\infty}^{\infty} dv_2^-\, F\big(v_2^-\big)\big(v_2^-\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^-\big)^{\alpha'\, k_3\cdot k_2} \\
&= \int_{-\infty}^{0} dv_2^-\, F\big(v_2^-\big)\big(v_2^-\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^-\big)^{\alpha'\, k_3\cdot k_2} + \int_{0}^{\infty} dv_2^-\, F\big(v_2^-\big)\big(v_2^-\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^-\big)^{\alpha'\, k_3\cdot k_2} \\
&= \big(e^{i\pi\alpha' k_1\cdot k_2} - e^{-i\pi\alpha' k_1\cdot k_2}\big)\int_{-\infty}^{0} dv_2^-\, F\big(v_2^-\big)\big(-v_2^-\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^-\big)^{\alpha'\, k_3\cdot k_2} \\
&= 2i\sin(\pi\alpha' k_1\cdot k_2)\widetilde{\mathcal{A}}_4(2,1,3,4)\,.
\end{aligned} \tag{4.21}$$

In the four-point case there is no sum in eq. (4.15) so we immediately find

$$\mathcal{M}_4 = -\frac{i}{2\pi\alpha'}\sin(\pi\alpha' k_1\cdot k_2)\mathcal{A}_4(1,2,3,4)\widetilde{\mathcal{A}}_4(2,1,3,4)\,. \tag{4.22}$$

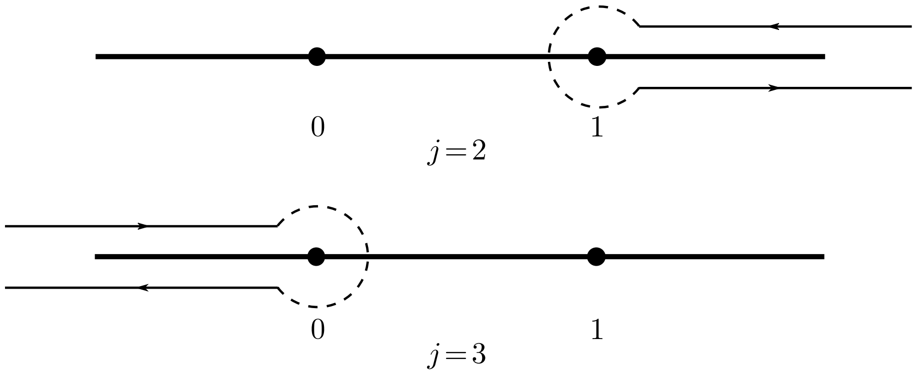


Figure 4.2: The two different ways of flipping contours in the four-point case.

If we instead pull the v_2^- contour to the right, *i.e.* $j=2$, we get

$$
\begin{aligned}
&\int_{-\infty}^{\infty} dv_2^- \, F\big(v_2^-\big)\big(v_2^-\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^-\big)^{\alpha'\, k_3\cdot k_2} \\
&= \int_{-\infty}^{1} dv_2^- \, F\big(v_2^-\big)\big(v_2^-\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^-\big)^{\alpha'\, k_3\cdot k_2} + \int_{1}^{\infty} dv_2^- \, F\big(v_2^-\big)\big(v_2^-\big)^{\alpha' k_1\cdot k_2}\big(1-v_2^-\big)^{\alpha'\, k_3\cdot k_2} \\
&= (-e^{-i\pi\alpha' k_3\cdot k_2} + e^{i\pi\alpha' k_3\cdot k_2}) \int_{1}^{\infty} dv_2^- \, F\big(v_2^-\big)\big(v_2^-\big)^{\alpha' k_1\cdot k_2}\big(v_2^- - 1\big)^{\alpha'\, k_3\cdot k_2} \\
&= 2i\sin(\pi\alpha' k_3\cdot k_2)\widetilde{\mathcal{A}}_4(1,3,2,4)\,, \qquad (4.23)
\end{aligned}
$$

and therefore

$$
\mathcal{M}_4 = -\frac{i}{2\pi\alpha'}\sin(\pi\alpha' k_3\cdot k_2)\mathcal{A}_4(1,2,3,4)\widetilde{\mathcal{A}}_4(1,3,2,4)\,. \qquad (4.24)
$$

The four-point case is especially simple, since there is only one integral to deform and the integrand takes such a simple form. We will now consider the five-point case to illustrate the general structure better.

4.2 Five-Point KLT-Relations

Starting with $j=4$, pulling the contour for v_2^- to the left, writing only the piece involving v_2^-, we get

$$
\begin{aligned}
&\int_{C_2} dv_2^- \, (v_2^-)^{\alpha' k_1\cdot k_2}(1-v_2^-)^{\alpha' k_4\cdot k_2}\,(v_3^- - v_2^-)^{\alpha' k_3\cdot k_2}\,F(v_2^-) \\
&= (e^{i\pi\alpha' k_1\cdot k_2} - e^{-i\pi\alpha' k_1\cdot k_2}) \int_{-\infty}^{0} dv_2^- \, (-v_2^-)^{\alpha' k_1\cdot k_2}(1-v_2^-)^{\alpha' k_4\cdot k_2}\,(v_3^- - v_2^-)^{\alpha' k_3\cdot k_2}\,F(v_2^-) \\
&= 2i\sin(\pi\alpha' k_1\cdot k_2) \int_{-\infty}^{0} dv_2^- \, (-v_2^-)^{\alpha' k_1\cdot k_2}(1-v_2^-)^{\alpha' k_4\cdot k_2}\,(v_3^- - v_2^-)^{\alpha' k_3\cdot k_2}\,F(v_2^-)\,. \qquad (4.25)
\end{aligned}
$$

Once again we write $(1-v_2^-)^{\alpha' k_4\cdot k_2}$ instead of $(v_2^- - 1)^{\alpha' k_4\cdot k_2}$ and $(v_3^- - v_2^-)^{\alpha' k_3\cdot k_2}$ instead of $(v_2^- - v_3^-)^{\alpha' k_3\cdot k_2}$ in order to compensate for the same change of order in the v_i^+ integrals of eq. (4.16). Now, as illustrated in the bottom of figure 4.3, we close the contour for v_3^-

to the left as well, and find, for the part involving the v_3^- variable,

$$
\begin{aligned}
&\int_{C_3} dv_3^-\,(v_3^-)^{\alpha' k_1\cdot k_3}(1-v_3^-)^{\alpha' k_4\cdot k_3}(v_3^- - v_2^-)^{\alpha' k_3\cdot k_2}\,F(v_3^-)\\
&= (e^{i\pi\alpha'(k_1+k_2)\cdot k_3} - e^{-i\pi\alpha'(k_1+k_2)\cdot k_3})\int_{-\infty}^{v_2^-} dv_3^-\,(-v_3^-)^{\alpha' k_1\cdot k_3}(1-v_3^-)^{\alpha' k_4\cdot k_3}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(v_2^- - v_3^-)^{\alpha' k_2\cdot k_3}\,F(v_3^-)\\
&\quad + (e^{i\pi\alpha' k_1\cdot k_3} - e^{-i\pi\alpha' k_1\cdot k_3})\int_{v_2^-}^{0} dv_3^-\,(-v_3^-)^{\alpha' k_1\cdot k_3}(1-v_3^-)^{\alpha' k_4\cdot k_3}(v_3^- - v_2^-)^{\alpha' k_2\cdot k_3}\,F(v_3^-)\\
&= 2i\sin\big(\pi\alpha'(k_1+k_2)\cdot k_3\big)\int_{-\infty}^{v_2^-} dv_3^-\,(-v_3^-)^{\alpha' k_1\cdot k_3}(1-v_3^-)^{\alpha' k_4\cdot k_3}(v_2^- - v_3^-)^{\alpha' k_2\cdot k_3}\,F(v_3^-)\\
&\quad + 2i\sin(\pi\alpha' k_1\cdot k_3)\int_{v_2^-}^{0} dv_3^-\,(-v_3^-)^{\alpha' k_1\cdot k_3}(1-v_3^-)^{\alpha' k_4\cdot k_3}(v_3^- - v_2^-)^{\alpha' k_2\cdot k_3}\,F(v_3^-)\,. \quad (4.26)
\end{aligned}
$$

We see that the total integration over v_2^- and v_3^- correspond to color-ordered open-string amplitudes, which we will denote $\widetilde{\mathcal{A}}_5$. This is to distinguish them from those following from the v_i^+ part. The whole v_i^- contribution in $M_5^g(2,3)$ can thus be written as

$$
\begin{aligned}
\propto\ &\sin(\pi\alpha' k_1\cdot k_2)\sin\big(\pi\alpha'(k_1+k_2)\cdot k_3\big)\widetilde{\mathcal{A}}_5(3,2,1,4,5)\\
&+\sin(\pi\alpha' k_1\cdot k_2)\sin(\pi\alpha' k_1\cdot k_3)\widetilde{\mathcal{A}}_5(2,3,1,4,5)\,. \qquad (4.27)
\end{aligned}
$$

Together with eq. (4.16) for $n=5$, and eq. (4.15), we obtain the following relation between the five-point closed-string amplitude $\mathcal{M}_5$ and the color-ordered open-string amplitudes $\mathcal{A}_5$, $\widetilde{\mathcal{A}}_5$

$$
\begin{aligned}
\mathcal{M}_5 = \frac{-1}{4\pi^2\alpha'^2}\Big[&\sin(\pi\alpha' k_1\cdot k_2)\sin\big(\pi\alpha'(k_1+k_2)\cdot k_3\big)\mathcal{A}_5(1,2,3,4,5)\widetilde{\mathcal{A}}_5(3,2,1,4,5)\\
&+\sin(\pi\alpha' k_1\cdot k_2)\sin(\pi\alpha' k_1\cdot k_3)\mathcal{A}_5(1,2,3,4,5)\widetilde{\mathcal{A}}_5(2,3,1,4,5)\Big]\\
&+(2\leftrightarrow 3)\,. \qquad (4.28)
\end{aligned}
$$

If we take the other extreme, *i.e.* closing both contours to the right ($j=2$), we get

$$
2i\sin(\pi\alpha' k_4\cdot k_3)\int_{1}^{+\infty} dv_3^-\,(v_3^-)^{\alpha' k_1\cdot k_3}(v_3^- - 1)^{\alpha' k_4\cdot k_3}(v_3^- - v_2^-)^{\alpha' k_2\cdot k_3}\,F(v_3^-)\,, \qquad (4.29)
$$

for the v_3^- integration, and

$$
\begin{aligned}
&2i\sin(\pi\alpha' k_4\cdot k_2)\int_{1}^{v_3^-} dv_2^-\,(v_2^-)^{\alpha' k_1\cdot k_2}(v_2^- - 1)^{\alpha' k_4\cdot k_2}(v_3^- - v_2^-)^{\alpha' k_3\cdot k_2}\,F(v_2^-) \qquad (4.30)\\
&+2i\sin\big(\pi\alpha'(k_4+k_3)\cdot k_2\big)\int_{v_3^-}^{+\infty} dv_2^-\,(v_2^-)^{\alpha' k_4\cdot k_2}(v_2^- - 1)^{\alpha' k_4\cdot k_2}(v_2^- - v_3^-)^{\alpha' k_3\cdot k_2}\,F(v_2^-)\,,
\end{aligned}
$$

for the v_2^- integration, see the top case in figure 4.3, *i.e.*

$$
\begin{aligned}
\mathcal{M}_5 = \frac{-1}{4\pi^2\alpha'^2}\Big[&\sin(\pi\alpha' k_4\cdot k_3)\sin(\pi\alpha' k_4\cdot k_2)\mathcal{A}_5(1,2,3,4,5)\widetilde{\mathcal{A}}_5(1,4,2,3,5)\\
&+\sin(\pi\alpha' k_4\cdot k_3)\sin\big(\pi\alpha'(k_4+k_3)\cdot k_2\big)\mathcal{A}_5(1,2,3,4,5)\widetilde{\mathcal{A}}_5(1,4,3,2,5)\Big]\\
&+(2\leftrightarrow 3)\,. \qquad (4.31)
\end{aligned}
$$

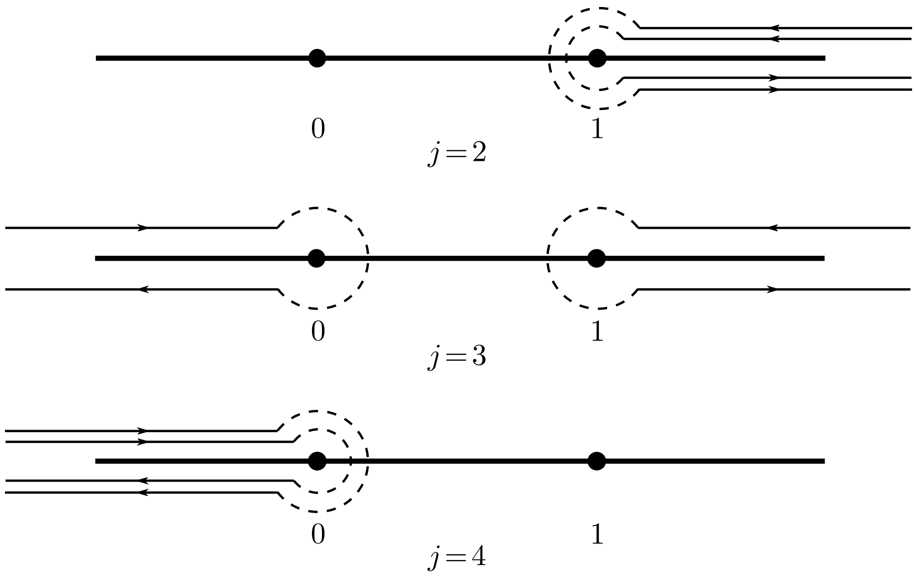


Figure 4.3: The three different ways of flipping contours in the five-point case.

Finally, we could have closed v_2^- to the left and v_3^- to the right, also illustrated in figure 4.3 ($j = 3$), resulting in

$$\begin{aligned}\mathcal{M}_5 = &\frac{-1}{4\pi^2\alpha'^2}\sin(\pi\alpha' k_1\cdot k_2)\sin(\pi\alpha' k_4\cdot k_3)\mathcal{A}_5(1,2,3,4,5)\widetilde{\mathcal{A}}_5(2,1,4,3,5)\\ &+(2\leftrightarrow 3)\,.\end{aligned} \tag{4.32}$$

These three forms can be nicely collected into one compact formula by introducing the *momentum kernel*

$$\mathcal{S}_{\alpha'}[i_1,\ldots,i_k|j_1,\ldots,j_k]_p \equiv (\pi\alpha'/2)^{-k}\prod_{t=1}^{k}\sin\big(\pi\alpha'\,(p\cdot k_{i_t}+\sum_{q>t}^{k}\theta(i_t,i_q)\,k_{i_t}\cdot k_{i_q})\big)\,, \tag{4.33}$$

where $\theta(i_t, i_q)$ equals 1 if the ordering of i_t and i_q is opposite in $\{i_1,\ldots,i_k\}$ and $\{j_1,\ldots,j_k\}$, and 0 if the ordering is the same. Here we defined $\mathcal{S}_{\alpha'}$ for a general number of legs, for instance

$$\begin{aligned}\mathcal{S}_{\alpha'}[2|2]_{k_1} &= (\pi\alpha'/2)^{-1}\sin\big(\pi\alpha' k_1\cdot k_2\big)\,,\\ \mathcal{S}_{\alpha'}[23|23]_{k_1} &= (\pi\alpha'/2)^{-2}\sin\big(\pi\alpha' k_1\cdot k_2\big)\sin\big(\pi\alpha' k_1\cdot k_3\big)\,,\\ \mathcal{S}_{\alpha'}[23|32]_{k_1} &= (\pi\alpha'/2)^{-2}\sin\big(\pi\alpha'(k_1+k_3)\cdot k_2\big)\sin\big(\pi\alpha' k_1\cdot k_3\big)\,,\end{aligned} \tag{4.34}$$

and so on. We will define $\mathcal{S}_{\alpha'}[\emptyset|\emptyset]_p = 1$ for empty sets. With this $\mathcal{S}_{\alpha'}$ function we can collect eq. (4.28), (4.31) and (4.32) into

$$\begin{aligned}\mathcal{M}_5 = &(-i/4)^2\times\\ &\sum_{\sigma}\sum_{\gamma,\beta}\mathcal{S}_{\alpha'}[\gamma(\sigma(2),\ldots,\sigma(j-1))|\sigma(2),\ldots,\sigma(j-1)]_{k_1}\mathcal{S}_{\alpha'}[\sigma(j),\ldots,\sigma(3)|\beta(\sigma(j),\ldots,\sigma(3))]_{k_4}\\ &\times\mathcal{A}_5(1,\sigma(2,3),4,5)\,\widetilde{\mathcal{A}}_5(\gamma(\sigma(2),\ldots,\sigma(j-1)),1,4,\beta(\sigma(j),\ldots,\sigma(3)),5)\,,\end{aligned} \tag{4.35}$$

with $j = \{2,3,4\}$. When $j = 2$ one should read the set $\{\sigma(2),\ldots,\sigma(j-1)\}$ as being empty, and likewise for $j = 4$ the set $\{\sigma(j),\ldots,\sigma(3)\}$ is empty.

4.3 General n-point KLT-Relations

With the four- and five-point examples in mind we see how the general case goes. Closing the contour for v_2^- to the left we find

$$\begin{aligned}
&\int_{C_2} dv_2^- \, (v_2^-)^{\alpha' k_1\cdot k_2}(1-v_2^-)^{\alpha' k_{n-1}\cdot k_2} \prod_{j=3}^{n-2}(v_j^- - v_2^-)^{\alpha' k_j\cdot k_2}\, F(v_2^-) \\
&= 2i\sin(\pi\alpha' k_1\cdot k_2)\int_{-\infty}^{0} dv_2^- \, (-v_2^-)^{\alpha' k_1\cdot k_2}(1-v_2^-)^{\alpha' k_{n-1}\cdot k_2} \prod_{j=3}^{n-2}(v_j^- - v_2^-)^{\alpha' k_j\cdot k_2}\, F(v_2^-)\,,
\end{aligned} \tag{4.36}$$

where we only show the part where v_2^- has branch points.

Continuing by closing the contour for v_3^- to the left as well leads to

$$\begin{aligned}
&\int_{C_3} dv_3^- \, (v_3^-)^{\alpha' k_1\cdot k_3}(1-v_3^-)^{\alpha' k_{n-1}\cdot k_3}(v_3^- - v_2^-)^{\alpha' k_2\cdot k_3} \prod_{j=4}^{n-2}(v_j^- - v_3^-)^{\alpha' k_j\cdot k_3}\, F(v_3^-) \\
&= 2i\sin(\pi\alpha' k_1\cdot k_3)\int_{v_2^-}^{0} dv_3^- \, (-v_3^-)^{\alpha' k_1\cdot k_3}(1-v_3^-)^{\alpha' k_{n-1}\cdot k_3}(v_3^- - v_2^-)^{\alpha' k_2\cdot k_3} \\
&\qquad\qquad \times \prod_{j=4}^{n-2}(v_j^- - v_3^-)^{\alpha' k_j\cdot k_3}\, F(v_3^-) \\
&+ 2i\sin\big(\pi\alpha'(k_1+k_2)\cdot k_3\big)\int_{-\infty}^{v_2^-} dv_3^- \, (-v_3^-)^{\alpha' k_1\cdot k_3}(1-v_3^-)^{\alpha' k_{n-1}\cdot k_3}(v_2^- - v_3^-)^{\alpha' k_2\cdot k_3} \\
&\qquad\qquad \times \prod_{j=4}^{n-2}(v_j^- - v_3^-)^{\alpha' k_j\cdot k_3}\, F(v_3^-)\,,
\end{aligned} \tag{4.37}$$

and so forth until we have closed the contour for v_{j-1}^- to the left.

When closing the contours to the right we start from the contour for v_{n-2}^- and go down to the one for v_j^-. Pulling the contour for v_{n-2}^- to the right gives

$$\begin{aligned}
&\int_{C_{n-2}} dv_{n-2}^- \, (v_{n-2}^-)^{\alpha' k_1\cdot k_{n-2}}(1-v_{n-2}^-)^{\alpha' k_{n-1}\cdot k_{n-2}} \prod_{j=2}^{n-3}(v_{n-2}^- - v_j^-)^{\alpha' k_j\cdot k_{n-2}}\, F(v_{n-2}^-) \\
&= 2i\sin(\pi\alpha' k_{n-1}\cdot k_{n-2})\int_{1}^{+\infty} dv_{n-2}^- \, (v_{n-2}^-)^{\alpha' k_1\cdot k_{n-2}}(v_{n-2}^- - 1)^{\alpha' k_{n-1}\cdot k_{n-2}} \\
&\qquad\qquad \times \prod_{j=2}^{n-3}(v_{n-2}^- - v_j^-)^{\alpha' k_j\cdot k_{n-2}}\, F(v_{n-2}^-)\,.
\end{aligned} \tag{4.38}$$

Similarly, closing the contour for v_{n-3}^- to the right

$$\begin{aligned}
\int_{C_{n-3}} dv_{n-3}^- \,(v_3^-)^{\alpha' k_1\cdot k_{n-3}}(1-v_{n-3}^-)^{\alpha' k_{n-1}\cdot k_{n-3}}(v_{n-2}^- - v_{n-3}^-)^{\alpha' k_{n-2}\cdot k_{n-3}}&\\
\times \prod_{j=2}^{n-4}(v_{n-3}^- - v_j^-)^{\alpha' k_j\cdot k_{n-3}}\,F(v_{n-3}^-)&\\
= 2i\sin(\pi\alpha' k_{n-1}\cdot k_{n-3})\int_1^{v_{n-2}^-} dv_{n-3}^-\,(v_{n-3}^-)^{\alpha' k_1\cdot k_{n-3}}(v_{n-3}^- -1)^{\alpha' k_{n-1}\cdot k_{n-3}}&\qquad (4.39)\\
\times (v_{n-2}^- - v_{n-3}^-)^{\alpha' k_{n-2}\cdot k_{n-3}}\prod_{j=2}^{n-4}(v_{n-3}^- - v_j^-)^{\alpha' k_j\cdot k_{n-3}}\,F(v_{n-3}^-)&\\
+ 2i\sin\big(\pi\alpha'(k_{n-1}+k_{n-2})\cdot k_{n-3}\big)\int_{v_{n-2}^-}^{+\infty} dv_{n-3}^-\,(v_{n-3}^-)^{\alpha' k_{n-1}\cdot k_{n-3}}(v_{n-3}^- -1)^{\alpha' k_{n-1}\cdot k_{n-3}}&\\
\times (v_{n-3}^- - v_{n-2}^-)^{\alpha' k_{n-2}\cdot k_{n-3}}\prod_{j=2}^{n-4}(v_{n-3}^- - v_j^-)^{\alpha' k_j\cdot k_{n-3}}\,F(v_{n-3}^-)\,,&
\end{aligned}$$

and so on until we reach the contour for v_j^-. The integrals over the v_i^- variables make up the color-ordered open-string amplitudes $\widetilde{\mathcal{A}}_n(\gamma(2,\ldots,j-1),1,n-1,\beta(j,\ldots,n-2),n)$.

Collecting all terms together, the expression for the v^- part of the integral in (4.12) takes the form

$$\begin{aligned}
(-i/4)^{n-3}\sum_\gamma\sum_\beta \mathcal{S}_{\alpha'}[\gamma(2,\ldots,j-1)|2,\ldots,j-1]_{k_1}\,\mathcal{S}_{\alpha'}[\beta(j,\ldots,n-2)|j,\ldots,n-2]_{k_{n-1}}&\\
\times\widetilde{\mathcal{A}}_n(\gamma(2,\ldots,j-1),1,n-1,\beta(j,\ldots,n-2),n)\,.&\qquad (4.40)
\end{aligned}$$

The full closed-string amplitude (4.1) is then obtained by multiplying the $\mathcal{A}_n$ amplitude, made up of the v^+ integrations in (4.16), with the contribution in (4.40), and sum over all orderings to get

$$\begin{aligned}
\mathcal{M}_n =\ & (-i/4)^{n-3}\times\\
&\sum_\sigma\sum_{\gamma,\beta}\mathcal{S}_{\alpha'}[\gamma(\sigma(2),\ldots,\sigma(j-1))|\sigma(2,\ldots,j-1)]_{k_1}\mathcal{S}_{\alpha'}[\sigma(j,\ldots,n-2)|\beta(\sigma(j),\ldots,\sigma(n-2))]_{k_{n-1}}\\
&\times\mathcal{A}_n(1,\sigma(2,\ldots,n-2),n-1,n)\,\widetilde{\mathcal{A}}_n(\gamma(\sigma(2),\ldots,\sigma(j-1)),1,n-1,\beta(\sigma(j),\ldots,\sigma(n-2)),n)\,,
\end{aligned}\qquad (4.41)$$

with $2\le j\le n-1$.

Expression (4.41) shows how to write an n-point closed-string amplitude $\mathcal{M}_n$ as the product of n-point color-ordered open-string amplitudes $\mathcal{A}_n$ and $\widetilde{\mathcal{A}}_n$, “glued” together by kinematic factors contained in the $\mathcal{S}_{\alpha'}$ function. The expression is a sum over $(n-3)!\times(j-2)!\times(n-1-j)!$ terms, taking its maximum value $(n-3)!\times(n-3)!$ for $j=2$ and $j=n-1$, and its minimum $(n-3)!\times(\lceil\frac{n}{2}\rceil-2)!\times(\lfloor\frac{n}{2}\rfloor-1)!$ for $j=\lceil n/2\rceil$[1].

Although it is one of the expressions with most terms, for $j=n-1$ the relation takes a particularly nice n-point form

$$\begin{aligned}
\mathcal{M}_n = (-i/4)^{n-3}\sum_{\sigma,\gamma}\mathcal{S}_{\alpha'}[\gamma(2,\ldots,n-2)|\sigma(2,\ldots,n-2)]_{k_1}&\\
\times\mathcal{A}_n(1,\sigma(2,\ldots,n-2),n-1,n)\widetilde{\mathcal{A}}_n(n-1,n,\gamma(2,\ldots,n-2),1)\,,&\qquad (4.42)
\end{aligned}$$

[1]The floor and ceiling functions are defined on half-integers as follows: $\lfloor n/2\rfloor=(n-1)/2$ if n is odd, or $n/2$ if n is even. $\lceil n/2\rceil=(n+1)/2$ if n is odd, or $n/2$ if n is even.

involving one $\mathcal{S}_{\alpha'}$ function. In addition it looks more symmetric in the sums over different ordered $\mathcal{A}_n$ and $\widetilde{\mathcal{A}}_n$ amplitudes. An equally nice form can be obtained with $j = 2$.

Notice that the right-hand side of eq. (4.41) or (4.42) does not look permutation invariant in all legs. However, we know that this must be the case since we started out from a closed-string amplitude which is totally permutation invariant. Although this is not immediately obvious from eq. (4.1), because of the fixing of the three points $z_1 = 0$, $z_{n-1} = 1$ and $z_n = \infty$. Likewise we found a freedom in the j-value which gives several different looking, but equivalent expressions. As one might have expected after the previous chapter, it is precisely the monodromy relations that save the permutational invariance of eq. (4.41), and the freedom of j. We will dwell more on this issue in the next chapter.

Chapter 5

KLT-Relations in Field Theory

The relations obtained in eq. (4.41) followed from factorization of closed-string amplitudes into a sum of products between open-string amplitudes. As such it is a relation satisfied to all orders in α', and in particular also in the field-theory limit $\alpha' \to 0$. In this limit the amplitudes go to their corresponding field theory expressions

$$\mathcal{M}_n \longrightarrow M_n\,, \qquad \mathcal{A}_n \longrightarrow A_n\,, \qquad \widetilde{\mathcal{A}}_n \longrightarrow \widetilde{A}_n\,, \tag{5.1}$$

and $\mathcal{S}_{\alpha'} \longrightarrow \mathcal{S}_0 \equiv \mathcal{S}$, where it follows from eq. (4.33) that

$$\mathcal{S}[i_1,\ldots,i_k|j_1,\ldots,j_k]_p = \prod_{t=1}^{k}\big(s_{pi_t} + \sum_{q>t}^{k}\theta(i_t,i_q)\,s_{i_t i_q}\big)\,. \tag{5.2}$$

Here $s_{ij} \equiv (k_i+k_j)^2 = 2k_i\cdot k_j$, or more generally $s_{ij\ldots k} = (p_i + p_j + \cdots + p_k)^2$, which will be used later on. Let us at this point make a slight change in the overall constant such that it is consistent with the normalization we used for helicity amplitudes in section 2.3. The n-point KLT-relations in field theory then take the final form

$$\begin{aligned} M_n &= (-1)^{n+1}\times \\ &\sum_{\sigma}\sum_{\gamma,\beta}\mathcal{S}[\gamma(\sigma(2),\ldots,\sigma(j-1))|\sigma(2,\ldots,j-1)]_{k_1}\mathcal{S}[\sigma(j),\ldots,\sigma(n-2)|\beta(\sigma(j,\ldots,n-2))]_{k_{n-1}} \\ &\times A_n(1,\sigma(2,\ldots,n-2),n-1,n)\,\widetilde{A}_n(\gamma(\sigma(2),\ldots,\sigma(j-1)),1,n-1,\beta(\sigma(j),\ldots,\sigma(n-2)),n)\,, \end{aligned} \tag{5.3}$$

where again we have the freedom of choosing $2 \leq j \leq n-1$.

Until now we have not been very explicit about the generality of these relations, *i.e.* for which gravity and gauge-theory amplitudes they are valid. However, since the above string derivation did not depend on the F and G functions, we should expect them to be satisfied for rather generic classes of amplitudes. In chapter 6 we return to this issue and examine how the KLT-relations map between different supersymmetric gravity and Yang-Mills theories. At the end of this chapter we will also see how the full tree-level gauge-theory amplitudes can be formulated as KLT-products between color-ordered gauge-theory and color-scalar amplitudes [66, 67]. For simplicity, we will in most of this chapter just think of it as a relation between graviton and gluon amplitudes.

Since it will be relevant below, and because of their simple expressions, let us write out the forms one obtain with $j = n-1$ and $j = 2$ explicitly, and at the same time introduce

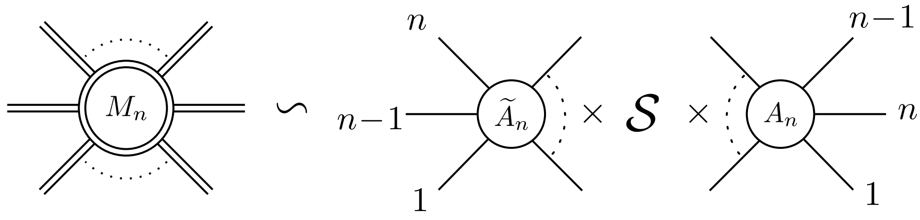


Figure 5.1: Diagrammatic representation of the KLT-relation with $j = n - 1$.

some short-hand notation. For $j = n - 1$ eq. (5.3) becomes

$$M_n = (-1)^{n+1} \sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} \widetilde{A}_n(n-1, n, \widetilde{\sigma}_{2,n-2}, 1)\mathcal{S}[\widetilde{\sigma}_{2,n-2}|\sigma_{2,n-2}]_{k_1} A_n(1, \sigma_{2,n-2}, n-1, n)\,, \tag{5.4}$$

and for $j = 2$ it is

$$M_n = (-1)^{n+1} \sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A_n(1, \sigma_{2,n-2}, n-1, n)\mathcal{S}[\sigma_{2,n-2}|\widetilde{\sigma}_{2,n-2}]_{k_{n-1}} \widetilde{A}_n(1, n-1, \widetilde{\sigma}_{2,n-2}, n)\,. \tag{5.5}$$

We have introduced the short-hand notation $\sigma_{2,n-2} \equiv \sigma(2, \ldots, n-2)$, for a σ-permutation of legs $2, \ldots, n-2$, and likewise for $\widetilde{\sigma}$. Note that σ and $\widetilde{\sigma}$ are two unrelated permutations, the tilde is just to remind us which one belongs to $\widetilde{A}_n$, and we are trying to be economic with our use of Greek letters. These notations for a permutation over a set of legs will be used interchangeably.

Like in the string-theory version of these relations, they are totally crossing symmetric in all legs, although they are only *manifest* symmetric in $2, 3, \ldots, n-2$. The crossing symmetry between, for instance, n and $n-1$ in eq. (5.4) can easily be seen as well, using the reflection symmetry $A_n(1, 2, \ldots, n) = (-1)^n A_n(n, n-1, \ldots, 1)$, and the identity

$$\mathcal{S}[i_1, \ldots, i_k|j_1, \ldots, j_k]_p = \mathcal{S}[j_k, \ldots, j_1|i_k, \ldots, i_1]_p\,. \tag{5.6}$$

This identity can be inferred from the definition of $\mathcal{S}$. The crossing symmetry between an arbitrary pair of legs involving 1, n or $n-1$ is not obvious at all.

Although the calculations in chapter 4 were a bit involved, at least one had an intuitive picture of breaking up a closed string into two open strings glued together by phase factors. It was also clear how to get different expressions for the factorization by choosing different kinds of closures for the contours. The field-theory limit then follows naturally when taking $\alpha' \to 0$. However, it would be quite unsatisfactory if we could not understand this expression without going through string theory first. This will be the main focus in rest of this chapter; how to see the KLT-relations from a purely field theoretical point of view, including the freedom of going between different expressions without having contours to deform. In order to get a better feel for the task that lies ahead let us start by looking at some lower-point examples and make some comments.

5.1 Lower-Point Examples

In section 2.3 we calculated the three-point gluon and graviton amplitudes explicitly using Feynman rules, and saw that they were related through the squaring relation

$$M_3(1, 2, 3) = A_3(1, 2, 3)\widetilde{A}_3(1, 2, 3)\,. \tag{5.7}$$

With the definition of $\mathcal{S}[\emptyset|\emptyset]_p = 1$ this is also contained in the above KLT-relations.

For four points eq. (5.4) reads

$$M_4(1,2,3,4) = -s_{12}A_4(1,2,3,4)\widetilde{A}_4(1,2,4,3)\,. \tag{5.8}$$

Compared to eq. (5.7) we see the appearance of a kinematic factor making sure to cancel one of the s_{12} poles present in both gauge-theory amplitudes. We also see that the total crossing symmetry of the right-hand side has already been well hidden.

When going to five points, expressed in the form with fewest terms ($j = 3$), we have

$$M_5 = s_{12}s_{34}A_5(1,2,3,4,5)\widetilde{A}_5(2,1,4,3,5) + s_{13}s_{24}A_5(1,3,2,4,5)\widetilde{A}_5(3,1,4,2,5)\,. \tag{5.9}$$

Here the total crossing symmetry of the right-hand side is by no means obvious, and even the correct cancellation of double-poles begins to get complicated to see. These properties get more and more disguised as we increase the number of external legs.

From these few examples it seems almost impossible to identify, to arbitrary multiplicity, the right-hand side of eq. (5.3) with a gravity amplitude taking a purely field theoretical/analytical point of view. Even the simplest features of gravity amplitudes have become very non-trivial statements on the gauge-theory side.

Before we can give a general field-theory proof of eq. (5.3) we need to make some preparations. This does not only lead to a better understanding of KLT-relations, but also introduce structures that are interesting in their own right. The first issue we will address is the matter of j-independence in eq. (5.3). To establish this from pure field theory we rephrase the BCJ-relations and show how they imply the freedom in j. Second, we introduce a rather unusual way of expressing the KLT-relations. This formula will have a higher degree of manifest crossing symmetry, compared to eq. (5.3), but it requires an off-shell regularization. Since it will be important in the proof of eq. (5.3), we need to prove this formula first. Finally, we need to investigate what happens with the right-hand side of eq. (5.3) when A_n and $\widetilde{A}_n$ belong to different helicity sectors.

5.2 BCJ-Relations Reexpressed

The BCJ-relations are closely connected to the momentum kernel in eq. (5.2). They can be rephrased in terms of the $\mathcal{S}$ function as

$$0 = \sum_{\sigma\in S_{n-2}} \mathcal{S}[\beta(2,\ldots,n-1)|\sigma(2,\ldots,n-1)]_{k_1} A_n(1,\sigma(2,\ldots,n-1),n)\,, \tag{5.10}$$

where β is just some arbitrary permutation of $2,3,\ldots,n-1$. This is satisfied due to BCJ-relations in the form of eq. (3.27). To see this, let us write out the five-point case with $\beta(2,3,4) = (2,3,4)$, in a suggestive way

$$\begin{aligned}
&\sum_{\sigma\in S_3} \mathcal{S}[2,3,4|\sigma(2,3,4)]_{k_1} A_5(1,\sigma(2,3,4),5)\\
&= \sum_{\sigma\in OP(\{2\}\cup\{3,4\})} \mathcal{S}[2,3,4|\sigma(2,3,4)]_{k_1} A_5(1,\sigma(2,3,4),5) \qquad (5.11)\\
&\quad + \sum_{\sigma\in OP(\{2\}\cup\{4,3\})} \mathcal{S}[2,3,4|\sigma(2,3,4)]_{k_1} A_5(1,\sigma(2,3,4),5)\,. \qquad (5.12)
\end{aligned}$$

The expression in (5.11) is given by

$$s_{13}s_{14}\big[s_{12}A_5(1,2,3,4,5) + (s_{12}+s_{23})A_5(1,3,2,4,5) + (s_{12}+s_{23}+s_{24})A_5(1,3,4,2,5)\big]\,, \tag{5.13}$$

which is zero, since the expression inside $[\cdots]$ is just a BCJ-relation in the form of eq. (3.27). Likewise (5.12) is

$$s_{14}(s_{13}+s_{34})\big[s_{12}A_5(1,2,4,3,5)+(s_{12}+s_{24})A_5(1,4,2,3,5) \\ +(s_{12}+s_{23}+s_{24})A_5(1,4,3,2,5)\big]\,, \tag{5.14}$$

and therefore also vanish due to eq. (3.27).

The general argument is as follows; divide the sum of $\sigma(2,\ldots,n-1)$ into a sum of groups where all, except the first leg in the β ordering, call it $\beta(2)$, have fixed ordering, and then insert $\beta(2)$ at any place. For each group all factors from $\mathcal{S}$ will be the same except for the factor contributing from $\beta(2)$. This leads to an expression in the form of eq. (3.27) and thereby vanish. In the above five-point example we had $\beta(2)=2$ and the two groups we summed over were those with ordering $\{3,4\}$ and $\{4,3\}$, respectively. In this way eq. (5.10) is a direct consequence of eq. (3.27).

The j-independence

We can now address the issue of the j-independence in eq. (5.3). Indeed, it will prove useful to first establish that all KLT-expressions, only differing in the j-value chosen, are equivalent. In this way, if we can prove eq. (5.3) for just one choice of j, we have proven them all.

To see the j-independence we need yet another rephrasing of the BCJ-relations, namely

$$\sum_{\alpha,\beta}\mathcal{S}[\alpha_{i_2,i_j}|i_2,\ldots,i_j]_{p_1}\mathcal{S}[i_{j+1},\ldots,i_{n-2}|\beta_{i_{j+1},i_{n-2}}]_{p_{n-1}}\widetilde{A}_n(\alpha_{i_2,i_j},1,n-1,\beta_{i_{j+1},i_{n-2}},n) \\ =\sum_{\alpha',\beta'}\mathcal{S}[\alpha'_{i_2,i_{j-1}}|i_2,\ldots,i_{j-1}]_{p_1}\mathcal{S}[i_j,i_{j+1},\ldots,i_{n-2}|\beta'_{i_j,i_{n-2}}]_{p_{n-1}}\widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,\beta'_{i_j,i_{n-2}},n)\,. \tag{5.15}$$

Although eq. (5.15) looks complicated it is straightforward to prove using only BCJ-relations in the form of eq. (3.27) and momentum conservation. It shows how to increase/decrease the number of legs in the α and β sets one leg at a time. The proof can be found in appendix B, as well as in [23], but it is not very enlightening. Instead we here illustrate how to use eq. (5.15) in the simple case of five points.

For $n=5$ and $j=2$ eq. (5.15) reads (where we have taken $i_k=k$)

$$\mathcal{S}[2|2]_{p_1}\mathcal{S}[3|3]_{p_4}\widetilde{A}_5(2,1,4,3,5)=\sum_{\beta'}\mathcal{S}[2,3|\beta'(2,3)]_{p_4}\widetilde{A}_5(1,4,\beta'(2,3),5)\,. \tag{5.16}$$

Since eq. (5.3) for $n=5$ and $j=2$ is

$$M_5=\sum_\sigma\sum_\beta A_5(1,\sigma(2,3),4,5)\mathcal{S}[\sigma(2,3)|\beta(2,3)]_{k_4}\widetilde{A}_5(1,4,\beta(2,3),5) \\ =\sum_\sigma A_5(1,\sigma(2,3),4,5)\sum_\beta\mathcal{S}[\sigma(2,3)|\beta(2,3)]_{k_4}\widetilde{A}_5(1,4,\beta(2,3),5)\,, \tag{5.17}$$

using eq. (5.16) we immediately get

$$M_5=\sum_\sigma A_5(1,\sigma(2,3),4,5)\mathcal{S}[\sigma(2)|\sigma(2)]_{p_1}\mathcal{S}[\sigma(3)|\sigma(3)]_{p_4}\widetilde{A}_5(\sigma(2),1,4,\sigma(3),5)\,, \tag{5.18}$$

which is eq. (5.3) with $j = 3$. Likewise, using eq. (5.15) with $j = 3$ we can rewrite eq. (5.18) into eq. (5.3) with $j = 4$.

This procedure generalizes to n points; by repeated use of eq. (5.15) the j-independence of eq. (5.3) follows.

It is interesting to note that the BCJ-relations could have been discovered much earlier. Equating different expressions for the KLT-relations lead to pure gauge-theory amplitude relations, which, as we have just seen, are directly related to the BCJ-relations. This was already apparent in [19], but at that time not really appreciated. Writing both the KLT- and BCJ-relations in terms of the $\mathcal{S}$ function, we see how closely connected they really are.

5.3 Regularized KLT-Relation and Soft-Limit Behaviour

Considering the scattering of n particles with momenta p_i, we introduce the following deformation of p_1 and p_n

$$\begin{aligned} p_1 &\longrightarrow p'_1 = p_1 - xq\,, \\ p_n &\longrightarrow p'_n = p_n + xq\,, \end{aligned} \tag{5.19}$$

where x is just some arbitrary parameter, and q is a four-vector satisfying $q \cdot p_1 = q^2 = 0$ and $q \cdot p_n \neq 0$. This preserves conservation of momentum and keeps $p_1'^2 = 0$, but makes $p_n'^2 = s_{1'2\ldots(n-1)} \neq 0$. As such it is an off-shell deformation of leg n.

The gravity amplitude M_n can then be obtained as the on-shell limit of [22]

$$M_n = (-1)^n \lim_{x\to 0} \sum_{\sigma,\widetilde{\sigma}\in S_{n-2}} \frac{\widetilde{A}_n(n',\widetilde{\sigma}_{2,n-1},1')\mathcal{S}[\widetilde{\sigma}_{2,n-1}|\sigma_{2,n-1}]_{p'_1} A_n(1',\sigma_{2,n-1},n')}{s_{1'2\ldots(n-1)}}\,. \tag{5.20}$$

As $x \to 0$ the denominator goes to zero, but due to eq. (5.10), so does the numerator. However, the total expression has a limit which is exactly equal to a gravity amplitude. We will later prove this statement, but in this section we instead show how eq. (5.20) is related to the soft-limit behaviour of eq. (5.4) [64], and compare it to the well-known soft-limit behaviour of gravity amplitudes [68, 69].

We emphasize that just like eq. (5.3) contained the correct three-point squaring relation when $n = 3$, so does eq. (5.20)

$$\begin{aligned} M_3 = &-\lim_{x\to 0} \frac{\widetilde{A}_3(3',2,1')s_{1'2}A_3(1',2,3')}{s_{1'2}} = -\lim_{x\to 0} \widetilde{A}_3(3',2,1')A_3(1',2,3') \\ &= \widetilde{A}_3(1,2,3)A_3(1,2,3)\,. \end{aligned} \tag{5.21}$$

Note that if we make a deformation with p'_1 being off-shell instead of p'_n, we can write the “dual” expression to eq. (5.20)

$$M_n = (-1)^n \lim_{y\to 0} \sum_{\sigma,\widetilde{\sigma}\in S_{n-2}} \frac{A_n(1',\sigma_{2,n-1},n')\mathcal{S}[\sigma_{2,n-1}|\widetilde{\sigma}_{2,n-1}]_{p'_n}\widetilde{A}_n(n',\widetilde{\sigma}_{2,n-1},1')}{s_{23\ldots n'}}\,, \tag{5.22}$$

where we have called the deformation parameter y.

Soft Limit of Gluon and Graviton Amplitudes

For the discussion in this section we assume leg n to be soft and having positive helicity. The soft-limit behaviour of color-ordered gluon amplitudes is [70]

$$\lim_{k_n^+\to 0} A_n(\cdots,a,n^+,b,\cdots) = S^{\mathrm{YM}}(a,n^+,b) \times A_{n-1}(\cdots,a,b,\cdots)\,, \tag{5.23}$$

with

$$S^{\mathrm{YM}}(a,n^+,b)=\frac{\langle ab\rangle}{\langle an\rangle\langle nb\rangle}\,. \tag{5.24}$$

Similar for graviton amplitudes [68, 69]

$$\lim_{k_n^+\to 0} M_n(\ldots,a,n^+,b,\ldots)=S^{\mathrm{gravity}}(n^+)\times M_{n-1}(\ldots,a,b,\ldots)\,, \tag{5.25}$$

where

$$S^{\mathrm{gravity}}(n^+)=-\frac{1}{\langle 1,n\rangle\langle n,n-1\rangle}\sum_{i=2}^{n-2}\frac{\langle 1,i\rangle\langle n-1,i\rangle[i,n]}{\langle n,i\rangle}\,. \tag{5.26}$$

Soft Limit of KLT-Relation

Start from the KLT-relation in eq. (5.4)

$$\begin{aligned}M_n=(-1)^{n+1}\sum_{\sigma,\gamma}&\mathcal{S}[\gamma(2,\ldots,n-2)|\sigma(2,\ldots,n-2)]_{k_1}\\&\times A_n(1,\sigma(2,\ldots,n-2),n-1,n^+)\widetilde{A}_n(n-1,n^+,\gamma(2,\ldots,n-2),1)\,,\end{aligned} \tag{5.27}$$

with the assumption that leg n has positive helicity. Taking the soft limit $k_n^+\to 0$ and using eq. (5.23) we get

$$\begin{aligned}\lim_{k_n^+\to 0}(-1)^{n+1}M_n\approx\sum_{\sigma,\gamma}&S^{\mathrm{YM}}(n-1,n^+,\gamma(2))S^{\mathrm{YM}}(n-1,n^+,1)\mathcal{S}[\gamma(2,\ldots,n-2)|\sigma(2,\ldots,n-2)]_{k_1}\\&\times A_{n-1}(1,\sigma(2,\ldots,n-2),n-1)\widetilde{A}_{n-1}(n-1,\gamma(2,\ldots,n-2),1)\,.\end{aligned} \tag{5.28}$$

We use "$\approx$" as a reminder that this is in the soft limit. The momentum kernel is not (explicitly) affected by this procedure. We see that

$$\begin{aligned}S^{\mathrm{YM}}(n-1,n^+,i)S^{\mathrm{YM}}(n-1,n^+,1)&=\frac{\langle n-1,1\rangle}{\langle n-1,n\rangle\langle n,1\rangle}\frac{\langle n-1,i\rangle}{\langle n-1,n\rangle\langle n,i\rangle}\\&=\frac{1}{s_{n,n-1}}\frac{\langle n-1,1\rangle[n,n-1]}{\langle n-1,n\rangle\langle n,1\rangle}\frac{\langle n-1,i\rangle}{\langle n,i\rangle}\,,\end{aligned} \tag{5.29}$$

and in the denominator $s_{n,n-1}=s_{12\ldots n-2}$. The soft limit of the n-point gravity amplitude can thereby be written as

$$\begin{aligned}&\lim_{k_n^+\to 0}(-1)^{n+1}M_n\approx\frac{\langle n-1,1\rangle[n,n-1]}{\langle n-1,n\rangle\langle n,1\rangle}\sum_{i=2}^{n-2}\frac{\langle n-1,i\rangle}{\langle n,i\rangle}\times\\&\sum_{\sigma,\gamma_i}\frac{\widetilde{A}_n(n-1,i,\gamma_i(2,..,n-2),1)\mathcal{S}[i,\gamma_i(2,..,n-2)|\sigma(2,..,n-2)]_{k_1}A_n(1,\sigma(2,..,n-2),n-1)}{s_{12\ldots n-2}}\,,\end{aligned} \tag{5.30}$$

where the permutation γ_i is over the $n-4$ legs $\{2,\ldots,i-1,i+1,\ldots,n-2\}$ with $2\leq i\leq n-2$.

Comparing with the soft limit $k_n^+\to 0$ of the graviton amplitude in eq. (5.25)

$$\lim_{k_n^+\to 0}M_n(1,2,\ldots,n-1,n)\approx\frac{1}{\langle n,1\rangle\langle n,n-1\rangle}\sum_{i=2}^{n-2}\frac{\langle n-1,i\rangle}{\langle n,i\rangle}\langle 1,i\rangle[i,n]\,M_{n-1}(1,2,\ldots,n-1)\,, \tag{5.31}$$

we identify

$$\sum_{\sigma,\gamma_i} \frac{\widetilde{A}_n(n{-}1,i,\gamma_i(2,..,n{-}2),1)\mathcal{S}[i,\gamma_i(2,..,n{-}2)|\sigma(2,..,n{-}2)]_{k_1}A_n(1,\sigma(2,..,n{-}2),n{-}1)}{s_{12\ldots n-2}}$$
$$\approx -\frac{\langle 1i\rangle[in]}{\langle 1,n-1\rangle[n-1,n]}(-1)^{n+1}M_{n-1}\,. \qquad (5.32)$$

Summing over i, using momentum conservation $\sum_{i=2}^{n-1}\langle 1i\rangle[in]=0$ and writing $(-1)^{n+1}=(-1)^{n-1}$, we find the expression in eq. (5.20) for $n-1$ points, *i.e.*

$$(-1)^{n-1}M_{n-1} \qquad (5.33)$$
$$\approx \sum_{\sigma,\gamma}\frac{\widetilde{A}_n(n{-}1,\gamma(2,\ldots,n{-}2),1)\mathcal{S}[\gamma(2,\ldots,n{-}2)|\sigma(2,\ldots,n{-}2)]_{k_1}A_n(1,\sigma(2,\ldots,n{-}2),n{-}1)}{s_{12\ldots n-2}}\,. \qquad (5.34)$$

Let us also mention that the connection between eq. (5.4) and (5.20) was shown more directly in [71]. Using

$$\frac{\langle 1i\rangle[in]}{\langle 1,n-1\rangle[n-1,n]}=\frac{s_{iq}}{s_{n-1,q}},\qquad \text{with}\qquad q\equiv|1\rangle[n|\,, \qquad (5.35)$$

it follows from the calculations in [71], that the above soft-limit procedure gives an equivalent description of eq. (5.34) as the off-shell regularization. The auxiliary momentum q satisfy all the requirements [22, 71] for an off-shell regularization, *i.e.* $q^2=k_1\cdot q=0$ and $q\cdot k_{n-1}\neq 0$.

5.4 Vanishing Identities

In the proof of eq. (5.20) and (5.3) below, we will encounter expressions that have the same form as the right-hand side of these KLT-relations, but with A_n and $\widetilde{A}_n$ belonging to different helicity sectors. Such expressions turn out to vanish altogether.

To be more specific let us denote a n-point N^kMHV helicity subamplitude as A_n^k, *i.e.* A_n^k has $2{+}k$ negative helicity gluons, with $k\in\{0,1,\ldots,n{-}4\}$ for non-vanishing amplitudes. We are not interested in exactly which legs are of negative and positive helicity, just the helicity sector the amplitude belongs to. We then have [24, 25, 72]

$$0=\sum_{\sigma}\sum_{\gamma,\beta}\mathcal{S}[\gamma(\sigma_{2,j-1})|\sigma_{2,j-1}]_{k_1}\mathcal{S}[\sigma_{j,n-2}|\beta(\sigma_{j,n-2})]_{k_{n-1}}A_n^k(1,\sigma_{2,j-1},\sigma_{j,n-2},n{-}1,n)$$
$$\times\,\widetilde{A}_n^{k'}(\gamma(\sigma_{2,j-1}),1,n{-}1,\beta(\sigma_{j,n-2}),n)\,, \qquad (5.36)$$

whenever $k\neq k'$. This is similar to the right-hand side of eq. (5.3), written in our short-hand notation, with a mismatch in the helicities of the color-ordered amplitudes.

At four points these relations are trivial, in the sense that we always have at least one amplitude that vanishes all by itself, however, at five points non-trivial cancellations start to appear. For instance, in the form with $j=3$, we have

$$0=s_{12}s_{34}A_5(1^-,2^-,3^+,4^+,5^+)\widetilde{A}_5(2^-,1^-,4^+,3^-,5^+)$$
$$+\,s_{13}s_{24}A_5(1^-,3^+,2^-,4^+,5^+)\widetilde{A}_5(3^-,1^-,4^+,2^-,5^+)\,. \qquad (5.37)$$

The vanishing identities are also valid when written in the form of eq. (5.20) or (5.22) with a regularization. They can be proven in the same way as we are gonna prove the KLT-relations below.

There is also a more physical understanding of the vanishing of such expressions. From a $\mathcal{N}_G = 8$ supergravity point of view the right-hand side of eq. (5.3), when A_n and $\widetilde{A}_n$ belong to different helicity sectors, corresponds to a $SU(8)$ R-symmetry violating gravity amplitude and must therefore vanish [26–28,73] (see also [51].) We will go into much more details about this in chapter 6.

5.5 Recursive Proof of KLT-Relations

We are now in a position to present the general proof of eq. (5.3) and (5.20) from a purely field theoretical point of view. Like the BCJ-relations in section 3.1 this will be given in terms of an induction proof. Since we already showed that the right-hand side of eq. (5.3) is equivalent for all j-values, we are free to choose any of the versions we like the most in proving eq. (5.3). For us this will be those written in eq. (5.4) and (5.5). In the proof of eq. (5.3) we will need (5.20), we therefore prove this formula first.

General Setup

Assume that we have checked eq. (5.3) and (5.20) up to $n-1$ points, *i.e.* that we have checked that the expressions on the right-hand sides are indeed equal to a gravity amplitude. We then write down the n-point expression for the right-hand side, call it R_n. Our goal is to show, only based on our knowledge of lower-point cases, that this is equal to the n-point gravity amplitude, that is $R_n = M_n$.

Similar to how the BCFW recursion relation was derived, we start out by deforming two momenta in our expression for R_n and consider the contour integral

$$B = \frac{1}{2\pi i}\oint \frac{dz}{z} R_n(z) = R_n(0) + \sum_{\text{poles } z_p \neq 0} \frac{\text{Res}_p(R_n(z), z_p)}{z_p}\,. \tag{5.38}$$

$R_n(0)$ is just the undeformed n-point expression and we have included a potential boundary term B on the left-hand side. Let us first argue that $B = 0$.

If we make a deformation in p_1 and p_n the $\mathcal{S}$ kernel in eq. (5.5) is independent of z. The vanishing of the boundary term is then guaranteed by the large-z behaviour of the gauge-theory amplitudes A_n and $\widetilde{A}_n$, this was also used in [67]. However, since we have already seen that all our KLT-expressions encoded in eq. (5.3) are equivalent, eq. (5.4) must have an equally well behaved large-z limit for such a (p_1, p_n)-shift.

In eq. (5.20) the $\mathcal{S}$ kernel contains an additional s_{ij} factor compared to eq. (5.4), but since eq. (5.20) also includes a numerator which will be affected by the deformation in p_1, these will balance out each other and again ensure a good behaviour in the large-z limit.

In the proof, presented below, we find it more convenient to make a deformation in p_1 and p_{n-1} when considering eq. (5.3). Such a deformation can not have a boundary term either, since we in section 5 already argued for the crossing symmetry between p_n and p_{n-1} in this expression.

With $B = 0$ established, the goal is to show that the sum of residues exactly make up a BCFW-expansion of a n-point gravity amplitude, and hence

$$M_n \sim \sum_{\text{poles } z_p \neq 0} \frac{\text{Res}_p(R_n(z), z_p)}{z_p}\,. \tag{5.39}$$

Regularized KLT-Relation

In this section we prove the regularized KLT-relation in eq. (5.20). The expression under consideration is

$$R_n = \lim_{x\to 0} R'_n(x)\,, \tag{5.40}$$

where we have defined

$$R'_n \equiv (-1)^n \sum_{\gamma,\beta} \frac{\widetilde{A}_n(n',\gamma_{2,n-1},1')\mathcal{S}[\gamma_{2,n-1}|\beta_{2,n-1}]_{p'_1} A_n(1',\beta_{2,n-1},n')}{s_{1'2\ldots n-1}}\,. \tag{5.41}$$

Here

$$p'_1 \equiv p_1 - xq\,, \qquad p'_n \equiv -p'_1 - p_2 \ldots - p_{n-1} = p_n + xq\,, \tag{5.42}$$

with $q\cdot p_1 = q^2 = 0$ and $q \cdot p_n \neq 0$, *i.e.*

$${p'_1}^2 = 0 \qquad \text{and} \qquad {p'_n}^2 \neq 0\,, \tag{5.43}$$

for $x \neq 0$. R'_n is therefore a completely well-defined object. In the on-shell limit we of course have

$$\lim_{x\to 0} p'_1 = p_1\,, \qquad \lim_{x\to 0} p'_n = p_n\,, \qquad \lim_{x\to 0} {p'_n}^2 = 0\,. \tag{5.44}$$

We then make a BCFW-shift in $(1',n)$, *i.e.*

$$\widehat{p}'_1 = p'_1 - z|1'\rangle[n|\,, \tag{5.45}$$

and

$$\widehat{p}'_n \equiv -\widehat{p}'_1 - p_2 \ldots - p_{n-1} = \underbrace{\left(p_n + z|1'\rangle[n|\right)}_{\widehat{p}_n} + xq\,, \tag{5.46}$$

still satisfying

$$\widehat{p}'^2_1 = 0 \qquad \text{and} \qquad \widehat{p}'^2_n \neq 0\,. \tag{5.47}$$

We need to examine the contributions from residues where $s_{\widehat{1}'2..k}$, for $k = 2,3,\ldots,n-2$, go on-shell. For each pole there are two different cases to consider:

- (A) The pole appears in only *one* of the amplitudes $\widetilde{A}_n$ or A_n.
- (B) The pole appears in *both* $\widetilde{A}_n$ and A_n.

We begin with case (A), and only consider the situation in which the pole shows up in $\widetilde{A}_n$. The case where the pole only appears in the A_n amplitude can be handled in a completely similar way. The residue can be calculated from

$$-\frac{1}{s_{1'2..k}} \lim_{z\to z_{1'2..k}} \left[s_{\widehat{1}'2..k}(z) R'_n(z)\right]\,. \tag{5.48}$$

The terms that can potentially contribute in (A) must all involve an $\widetilde{A}_n$ amplitude with the set of legs $\{1', 2, \ldots, k\}$ next to each other, *i.e.* the residue is calculated from

$$\frac{1}{s_{1'2..k}} \lim_{z\to z_{1'2..k}} \left[s_{\widehat{1}'2..k} \sum_{\gamma,\beta} \frac{\widetilde{A}_n(\widehat{n}', \gamma_{2,n-1}, \widehat{1}')\mathcal{S}[\gamma_{2,n-1}|\beta_{2,n-1}]_{\widehat{p}_1'} A_n(\widehat{1}', \beta_{2,n-1}, \widehat{n}')}{s_{\widehat{1}'2...n-1}} \right]$$
$$= \frac{1}{s_{1'2..k}} \lim_{z\to z_{1'2..k}} \left[\frac{s_{\widehat{1}'2..k}}{s_{\widehat{1}'2...n-1}} \sum_{\gamma,\sigma,\beta} \widetilde{A}_n(\widehat{n}', \gamma_{k+1,n-1}, \sigma_{2,k}, \widehat{1}') \right.$$
$$\left. \times\, \mathcal{S}[\gamma_{k+1,n-1}\sigma_{2,k}|\beta_{2,n-1}]_{\widehat{p}_1'} A_n(\widehat{1}', \beta_{2,n-1}, \widehat{n}') \right], \tag{5.49}$$

where we have omitted the overall sign-factor $(-1)^{n+1}$, which can easily be reinstated into the proof. We stress that all β-permutations that would lead to a $s_{1'2...k}$-pole in A_n have been excluded. Following from the definition of $\mathcal{S}$ in eq. (5.2), we can write

$$\mathcal{S}[\gamma_{k+1,n-1}\sigma_{2,k}|\beta_{2,n-1}]_{\widehat{p}_1'} = \mathcal{S}[\sigma_{2,k}|\rho_{2,k}]_{\widehat{p}_1'} \times \mathcal{G}\,, \tag{5.50}$$

where ρ denotes the relative ordering of legs $2, 3, \ldots, k$ in the β set, and $\mathcal{G}$ is a factor which is independent of σ, but otherwise irrelevant for what follows.

Using the above decomposition, and the factorization property of tree amplitudes in eq. (2.70), we get

$$\frac{1}{s_{1'2..k}} \lim_{z\to z_{1'2..k}} \left[\frac{1}{s_{\widehat{1}'2...n-1}} \sum_{\gamma,\sigma,\beta} s_{\widehat{1}'2..k} \widetilde{A}_n(\widehat{n}', \gamma_{k+1,n-1}, \sigma_{2,k}, \widehat{1}')\mathcal{S}[\sigma_{2,k}|\rho_{2,k}]_{\widehat{p}_1'}\, \mathcal{G} A_n(\widehat{1}', \beta_{2,n-1}, \widehat{n}') \right]$$
$$= \frac{1}{s_{1'2..k}} \left[\frac{1}{s_{\widehat{1}'2...n-1}} \sum_{\gamma,\sigma,\beta} \left(\sum_h \widetilde{A}(\widehat{n}', \gamma_{k+1,n-1}, -\widehat{P}^h)\widetilde{A}(\widehat{P}^{-h}, \sigma_{2,k}, \widehat{1}') \right) \right.$$
$$\left. \times\, \mathcal{S}[\sigma_{2,k}|\rho_{2,k}]_{\widehat{p}_1'}\, \mathcal{G} A_n(\widehat{1}', \beta_{2,n-1}, \widehat{n}') \right]$$
$$= \frac{1}{s_{1'2..k}} \left[\frac{1}{s_{\widehat{1}'2...n-1}} \sum_{\gamma,\sigma,\beta} \left(\sum_h \widetilde{A}(\widehat{n}', \gamma_{k+1,n-1}, -\widehat{P}^h) \right.\right.$$
$$\left.\left. \times \left\{ \sum_\sigma \widetilde{A}(\widehat{P}^{-h}, \sigma_{2,k}, \widehat{1}')\mathcal{S}[\sigma_{2,k}|\rho_{2,k}]_{\widehat{p}_1'} \right\} \right) \mathcal{G} A_n(\widehat{1}', \beta_{2,n-1}, \widehat{n}') \right]. \tag{5.51}$$

Since $\widehat{p}_1', p_2, \ldots, p_k, \widehat{P}$ are now on-shell, it follows from eq. (5.10) that

$$\sum_\sigma \widetilde{A}(\widehat{P}^{-h}, \sigma_{2,k}, \widehat{1}')\mathcal{S}[\sigma_{2,k}|\rho_{2,k}]_{\widehat{p}_1'} = 0\,, \tag{5.52}$$

and thus the whole contribution vanish, even in the regularized expression.

We have just seen that part (A) will not contribute to the sum of residues. We should therefore expect part (B) to make up all the contributions by itself. Let us see how this comes about. We are examining terms in which both A_n and $\widetilde{A}_n$ have the $s_{1'2..k}$ pole. Once again we consider

$$-\frac{1}{s_{1'2..k}} \lim_{z\to z_{1'2..k}} \left[s_{\widehat{1}'2..k}(z) R_n'(z) \right], \tag{5.53}$$

but this time of the form

$$\frac{1}{s_{1'2..k}} \lim_{z\to z_{1'2..k}} \left[\frac{s_{\widehat{1}'2..k}}{s_{\widehat{1}'2...n-1}} \sum_{\sigma,\gamma,\delta,\beta} \widetilde{A}_n(\widehat{n}', \gamma_{k+1,n-1}, \sigma_{2,k}, \widehat{1}') \right.$$
$$\left. \times\, \mathcal{S}[\gamma_{k+1,n-1}\sigma_{2,k}|\delta_{2,k}\beta_{k+1,n-1}]_{\widehat{p}'_1} A_n(\widehat{1}', \delta_{2,k}, \beta_{k+1,n-1}, \widehat{n}') \right]. \tag{5.54}$$

In the limit where $\widehat{P} = \widehat{p}'_1 + p_2 + \cdots + p_k$ goes on-shell, we can write

$$\mathcal{S}[\gamma_{k+1,n-1}\sigma_{2,k}|\delta_{2,k}\beta_{k+1,n-1}]_{\widehat{p}'_1} = \mathcal{S}[\sigma_{2,k}|\delta_{2,k}]_{\widehat{p}'_1} \times \mathcal{S}[\gamma_{k+1,n-1}|\beta_{k+1,n-1}]_{\widehat{P}}\,, \tag{5.55}$$

i.e.

$$\frac{1}{s_{1'2..k}} \lim_{z\to z_{1'2..k}} \left[\frac{1}{s_{\widehat{1}'2...n-1}} \sum_{\sigma,\gamma,\delta,\beta} \left[\sum_h \widetilde{A}(\widehat{n}', \gamma_{k+1,n-1}, \widehat{P}^h) \widetilde{A}(-\widehat{P}^{-h}, \sigma_{2,k}, \widehat{1}') \right] \right.$$
$$\left. \times\, \mathcal{S}[\sigma_{2,k}|\delta_{2,k}]_{\widehat{p}'_1} \mathcal{S}[\gamma_{k+1,n-1}|\beta_{k+1,n-1}]_{\widehat{P}} \left[\sum_h \frac{A(\widehat{1}', \delta_{2,k}, -\widehat{P}^{-h}) A(\widehat{P}^h, \beta_{k+1,n-1}, \widehat{n}')}{s_{\widehat{1}'2..k}} \right] \right]. \tag{5.56}$$

Strictly speaking we can not factorize the $\mathcal{S}$ kernel and the amplitudes *before* taking the $z \to z_{1'2..k}$ limit, however, since this is how these quantities begin to factorize as we get very close to the limit we write it out like above. Eq. (5.56) can be rewritten as

$$\frac{1}{s_{1'2..k}} \lim_{z\to z_{1'2..k}} \left[\sum_h \left[\left(\sum_{\sigma,\delta} \frac{\widetilde{A}(-\widehat{P}^{-h}, \sigma_{2,k}, \widehat{1}') \mathcal{S}[\sigma_{2,k}|\delta_{2,k}]_{\widehat{p}'_1} A(\widehat{1}', \delta_{2,k}, -\widehat{P}^{-h})}{s_{\widehat{1}'2..k}} \right) \right. \right.$$
$$\left. \left. \times \left(\sum_{\gamma,\beta} \frac{\widetilde{A}(\widehat{n}', \gamma_{k+1,n-1}, \widehat{P}^h) \mathcal{S}[\gamma_{k+1,n-1}|\beta_{k+1,n-1}]_{\widehat{P}} A(\widehat{P}^h, \beta_{k+1,n-1}, \widehat{n}')}{s_{\widehat{1}'2..n-1}} \right) \right] \right]$$
$$+ \text{(mixed helicity terms)}\,, \tag{5.57}$$

where the "mixed helicity terms" are expressions of the exact same form as line one and two in (5.57), but with products between amplitudes with $(-\widehat{P}^h, -\widehat{P}^{-h})$ and $(\widehat{P}^h, \widehat{P}^{-h})$ instead of $(-\widehat{P}^h, -\widehat{P}^h)$ and $(\widehat{P}^{-h}, \widehat{P}^{-h})$, respectively.

The second $(\cdots)$-term above is safe from singularities in the limit $z = z_{1'2..k}$, since this does not take the (regularized) pole $s_{\widehat{1}'2..n-1} = s_{\widehat{P}k+1..n-1}$ to zero, so we write

$$\frac{1}{s_{1'2..k}} \sum_h \lim_{z\to z_{1'2..k}} \left[\left(\sum_{\sigma,\delta} \frac{\widetilde{A}(-\widehat{P}^{-h}, \sigma_{2,k}, \widehat{1}') \mathcal{S}[\sigma_{2,k}|\delta_{2,k}]_{\widehat{p}'_1} A(\widehat{1}', \delta_{2,k}, -\widehat{P}^{-h})}{s_{\widehat{1}'2..k}} \right) \right]$$
$$\times \left(\sum_{\gamma,\beta} \frac{\widetilde{A}(\widehat{n}', \gamma_{k+1,n-1}, \widehat{P}^h) \mathcal{S}[\gamma_{k+1,n-1}|\beta_{k+1,n-1}]_{\widehat{P}} A(\widehat{P}^h, \beta_{k+1,n-1}, \widehat{n}')}{s_{\widehat{P}k+1..n-1}} \right)$$
$$+ \text{(mixed helicity terms)}\,. \tag{5.58}$$

Recall that we are here working with the regularized expression, and to get the final result we should take the $x \to 0$ limit. Taking this limit on eq. (5.58) the first term is completely well-defined for $x = 0$, and we have

$$\begin{aligned}
&\sum_h \lim_{z\to z_{12..k}} \left[\sum_{\sigma,\delta} \frac{\widetilde{A}(-\widehat{P}^{-h}, \sigma_{2,k}, \widehat{1}) \mathcal{S}[\sigma_{2,k}|\delta_{2,k}]_{\widehat{p}'_1} A(\widehat{1}, \delta_{2,k}, -\widehat{P}^{-h})}{s_{\widehat{1}2..k}} \right] \\
&\times \frac{1}{s_{12..k}} \times \lim_{x\to 0} \left[\sum_{\gamma,\beta} \frac{\widetilde{A}(\widehat{n}', \gamma_{k+1,n-1}, \widehat{P}^{h}) \mathcal{S}[\gamma_{k+1,n-1}|\beta_{k+1,n-1}]_{\widehat{P}} A(\widehat{P}^{h}, \beta_{k+1,n-1}, \widehat{n}')}{s_{\widehat{P}k+1..n-1}} \right] \\
&+ (\text{mixed helicity terms})\,. \qquad (5.59)
\end{aligned}$$

In the first term $\widehat{p}_1$ is on-shell, as it was from the beginning, but $-\widehat{P}$ (here playing the role of leg "n") is only on-shell in the limit $z = z_{12..k}$. In the second term $\widehat{P}$ plays the role of leg "1" and is on-shell since $z = z_{1'2..k}$ in this expression, however, $\widehat{p}'_n$ is still off-shell until we set $x = 0$. We therefore have two regularized expressions with the same form as the original R_n, just for lower points, so from induction we conclude that

$$\sum_h \frac{M_{k+1}(\widehat{1}, 2, ..., -\widehat{P}^{-h}) M_{n-k+1}(\widehat{P}^{h}, k+1, ..., \widehat{n})}{s_{12..k}} + (\text{mixed helicity terms})\,. \qquad (5.60)$$

The "mixed helicity terms" are examples of the vanishing identities we introduced in section 5.4, and will not contribute at all. We can prove these vanishing identities by the same steps as above, just starting from a R_n expression where the amplitudes $\widetilde{A}_n^{k'}$ and A_n^k have $k' \neq k$. One always ends up with products like in eq. (5.59) where at least one of the expressions represents a lower-point vanishing identity and thereby get that all contributions from both (A) and (B) are zero. For the explicit derivation see [24].

What is left is exactly the BCFW-contribution to the n-point gravity amplitude from a $s_{12...k}$ pole. Due to the manifest $(n-2)!$ symmetry in R'_n all other contributions from poles related to these by a permutation of legs $2, 3, \ldots, n-1$ can then be obtained by the corresponding permutation in (5.60), *i.e.* we have

$$\begin{aligned}
R_n &= -\sum_{\text{poles } z_p\neq 0} \frac{\text{Res}_p(R_n(z), z_p)}{z_p} = \sum_{Perm.} \sum_h \frac{M_{k+1}(\widehat{1}, 2, ..., -\widehat{P}^{-h}) M_{n-k+1}(\widehat{P}^{h}, k+1, ..., \widehat{n})}{s_{12..k}} \\
&= M_n\,, \qquad (5.61)
\end{aligned}$$

which concludes the induction proof of eq. (5.20).

General KLT-Relation

The proof of eq. (5.3) will be done using the same method as above and there will be many similarities between the two, but also some steps which need to be treated a bit different. Once again we begin by examining the residues from poles of the form $s_{12...k}$, and we will be using R_n in the form of eq. (5.4).

First we consider (A), taking the pole to appear only in $\widetilde{A}_n$. The terms from eq. (5.4) that can potentially contribute are of the form

$$\sum_{\sigma,\widetilde{\sigma},\alpha} \widetilde{A}_n(\widehat{n-1}, n, \widetilde{\sigma}_{k+1,n-2}, \alpha_{2,k}, \widehat{1}) \mathcal{S}[\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}|\sigma_{2,n-2}]_{\widehat{p}_1} A_n(\widehat{1}, \sigma_{2,n-2}, \widehat{n-1}, n)\,, \quad (5.62)$$

where again we omit overall sign factors, which can easily be reinstated. Remember that we have excluded all σ-permutations that lead to a $s_{12...k}$ pole in A_n. From this we get the residue

$$\sum_{\sigma,\widetilde{\sigma},\alpha} \frac{\sum_h \widetilde{A}(\widehat{n-1}, n, \widetilde{\sigma}_{k+1,n-2}, \widehat{P}^h)\widetilde{A}(-\widehat{P}^{-h}, \alpha_{2,k}, \widehat{1})}{s_{12\ldots k}} \mathcal{S}[\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}|\sigma_{2,n-2}]_{\widehat{p}_1} \times A_n(\widehat{1}, \sigma_{2,n-2}, \widehat{n-1}, n)\,, \tag{5.63}$$

where $\widehat{P} \equiv \widehat{p}_1 + p_2 + \cdots + p_k$, and we used the factorization property of amplitudes in eq. (2.70). Note that the pole $s_{\widehat{1}2\ldots k}$, from the factorization of $\widetilde{A}_n$, has been replaced with $s_{12\ldots k}$, *i.e.* without the hat on 1, from the calculation of the residue. Like above we write

$$\mathcal{S}[\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}|\sigma_{2,n-2}]_{\widehat{p}_1} = \mathcal{S}[\alpha_{2,k}|\rho_{2,k}]_{\widehat{p}_1} \times (\text{a factor independent of } \alpha)\,, \tag{5.64}$$

where ρ denotes the relative ordering of legs $2, 3, \ldots, k$ in the σ set. Collecting everything in (5.63) that involves the α-permutation we get something of the form

$$\sum_{\sigma}\sum_{h} \underbrace{\left(\sum_{\alpha} \widetilde{A}(-\widehat{P}^{-h}, \alpha_{2,k}, \widehat{1})\mathcal{S}[\alpha_{2,k}|\rho_{2,k}]_{\widehat{p}_1}\right)}_{0} \times \sum_{\widetilde{\sigma}}[\cdots]\,. \tag{5.65}$$

The quantity inside $(\cdots)$ vanishes when $\widehat{P}$ is on-shell (to get it in the exact same form as eq. (5.10) use the reflection symmetry and eq. (5.6)). We again conclude that contributions from (A) vanish altogether, and move on to consider (B).

Since both $\widetilde{A}_n$ and A_n now contains the pole $s_{12\ldots k}$, they must both have the set of legs $\{1, 2, \ldots, k\}$ collected next to each other. The contributing terms, from eq. (5.4), then take the form

$$\sum_{\sigma,\widetilde{\sigma},\alpha,\beta} \widetilde{A}_n(\widehat{n-1}, n, \widetilde{\sigma}_{k+1,n-2}, \alpha_{2,k}, \widehat{1})\mathcal{S}[\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}|\beta_{2,k}\sigma_{k+1,n-2}]_{\widehat{p}_1} \times A_n(\widehat{1}, \beta_{2,k}, \sigma_{k+1,n-2}, \widehat{n-1}, n)\,. \tag{5.66}$$

Using that when $\widehat{P} = \widehat{p}_1 + p_2 + \cdots + p_k$ goes on-shell, we can write

$$\mathcal{S}[\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}|\beta_{2,k}\sigma_{k+1,n-2}]_{\widehat{p}_1} = \mathcal{S}[\alpha_{2,k}|\beta_{2,k}]_{\widehat{p}_1} \times \mathcal{S}[\widetilde{\sigma}_{k+1,n-2}|\sigma_{k+1,n-2}]_{\widehat{P}}\,, \tag{5.67}$$

the residue for $s_{12\ldots k}$ can be expressed as

$$\begin{aligned} &\frac{1}{s_{12\ldots k}} \sum_h \lim_{z\to z_{12\ldots k}} \sum_{\alpha,\beta} \frac{\widetilde{A}(-\widehat{P}^h, \alpha_{2,k}, \widehat{1})\mathcal{S}[\alpha_{2,k}|\beta_{2,k}]_{\widehat{p}_1} A(\widehat{1}, \beta_{2,k}, -\widehat{P}^h)}{s_{\widehat{1}2\ldots k}} \\ &\times \sum_{\sigma,\widetilde{\sigma}} \widetilde{A}(\widehat{n-1}, n, \widetilde{\sigma}_{k+1,n-2}, \widehat{P}^{-h})\mathcal{S}[\widetilde{\sigma}_{k+1,n-2}|\sigma_{k+1,n-2}]_{\widehat{P}} A(\widehat{P}^{-h}, \sigma_{k+1,n-2}, \widehat{n-1}, n) \\ &+ (\text{mixed helicity terms})\,. \end{aligned} \tag{5.68}$$

The "mixed helicity terms" are nothing but vanishing identities and can therefore be discarded. These identities, taking the form of eq. (5.36), can be proven by the same procedure as already discussed in the proof of the regularized KLT-relation. However, as evident from (5.68), they do require that one has first established the vanishing identities in the form of eq. (5.20), which was discussed above.

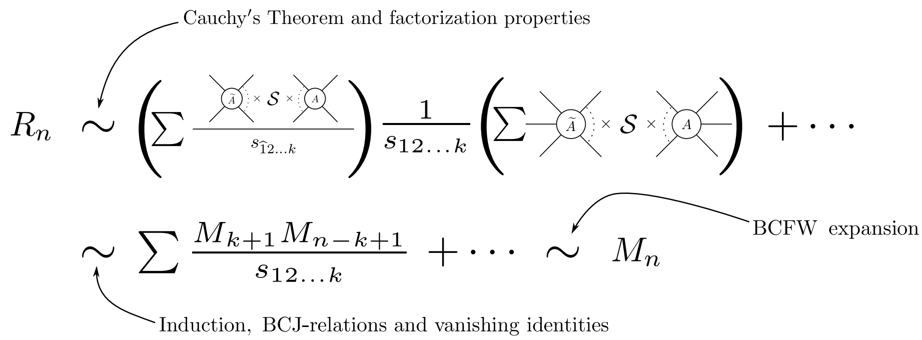


Figure 5.2: Schematic outline of field theory proof.

We are left with a sum over α and β, which precisely make up the regularized KLT-relation in eq. (5.20), *i.e.* it is just $M_{k+1}(\widehat{1},2,\ldots,k,-\widehat{P}^h)$, and a sum over σ and $\widetilde{\sigma}$ which is an $n-k+1$ point version of eq. (5.4) and hence, by induction, equal to $M_{n-k+1}(k+1,\ldots,\widehat{P}^{-h})$. Altogether (5.68) is

$$\sum_h \frac{M_{k+1}(\widehat{1},2,\ldots,k,-\widehat{P}^h)M_{n-k+1}(k+1,\ldots,\widehat{P}^{-h})}{s_{12\ldots k}}\,. \tag{5.69}$$

This is the BCFW-contribution to the n-point gravity amplitude from a $s_{12\ldots k}$ pole, and due to the manifest $(n-3)!$ symmetry we can immediately obtain the contributions for all poles related to these by a permutation of legs $2,3,\ldots,n-2$.

Contrary to the proof of eq. (5.20), we are still not completely done, since the above analysis does not cover the pole contributions involving both 1 and n, *i.e.* poles of the form $s_{12\ldots kn}=s_{k+1\ldots n-1}$. It was because of the larger manifest permutation symmetry in eq. (5.20), that all cases could be covered in one go. In order to investigate the missing contributions, we use the form given in eq. (5.5). It is well suited for this case since leg 1 and n are always next to each other here. As already mentioned, we are free to use whichever expression contained in eq. (5.3) we want to calculate a residue, they are just different ways of writing the same quantity. The following calculations are very similar to what we have already seen so we go through it a bit more briefly.

Part (A) of the residue for pole $s_{12\ldots kn}$ takes the form

$$\sum_{\sigma,\widetilde{\sigma},\alpha} A_n(\widehat{1},\sigma_{2,n-2},\widehat{n-1},n)\mathcal{S}[\sigma_{2,n-2}|\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}]_{\widehat{p}_{n-1}} \\ \times\sum_h \frac{\widetilde{A}(\widehat{n-1},\widetilde{\sigma}_{k+1,n-2},\widehat{P}^h)\widetilde{A}(-\widehat{P}^{-h},\alpha_{2,k},n,\widehat{1})}{s_{12\ldots kn}}\,, \tag{5.70}$$

where we assume the pole appears in the $\widetilde{A}_n$ amplitude. Using the factorization property

$$\mathcal{S}[\sigma_{2,n-2}|\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}]_{\widehat{p}_{n-1}} = \mathcal{S}[\rho_{k+1,n-2}|\widetilde{\sigma}_{k+1,n-2}]_{\widehat{p}_{n-1}}\times(\text{a factor independent of }\widetilde{\sigma})\,, \tag{5.71}$$

where ρ denotes the relative ordering of leg $k+1,\ldots,n-2$ in the σ set, we once again find that these contributions contain a factor of

$$\sum_{\widetilde{\sigma}} \widetilde{A}(\widehat{n-1},\widetilde{\sigma}_{k+1,n-2},\widehat{P}^h)\mathcal{S}[\rho_{k+1,n-2}|\widetilde{\sigma}_{k+1,n-2}]_{\widehat{p}_{n-1}} = 0\,, \tag{5.72}$$

that vanishes. There is therefore no contribution from (A).

Considering part (B) for the $s_{12\ldots kn}$ pole, the contributing terms from eq. (5.5) are

$$\sum_{\sigma,\widetilde{\sigma},\alpha,\beta} A_n(\widehat{1},\beta_{2,k},\sigma_{k+1,n-2},\widehat{n-1},n)\mathcal{S}[\beta_{2,k}\sigma_{k+1,n-2}|\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}]_{\widehat{p}_{n-1}} \times \widetilde{A}_n(\widehat{1},\widehat{n-1},\widetilde{\sigma}_{k+1,n-2},\alpha_{2,k},n)\,, \tag{5.73}$$

with $\mathcal{S}$ satisfying the factorization property (when $\widehat{P}$ goes on-shell)

$$\mathcal{S}[\beta_{2,k}\sigma_{k+1,n-2}|\widetilde{\sigma}_{k+1,n-2}\alpha_{2,k}]_{\widehat{p}_{n-1}} = \mathcal{S}[\sigma_{k+1,n-2}|\widetilde{\sigma}_{k+1,n-2}]_{\widehat{p}_{n-1}} \times \mathcal{S}[\beta_{2,k}|\alpha_{2,k}]_{\widehat{P}}\,. \tag{5.74}$$

Hence the residue can be written

$$\frac{1}{s_{12\ldots kn}}\sum_h\sum_{\alpha,\beta} A(\widehat{1},\beta_{2,k},\widehat{P}^h,n)\mathcal{S}[\beta_{2,k}|\alpha_{2,k}]_{\widehat{P}}\widetilde{A}(\widehat{1},\widehat{P}^h,\alpha_{2,k},n)\ \times$$
$$\lim_{z\to z_{k+1\ldots n-1}}\sum_{\widetilde{\sigma},\sigma}\frac{A(-\widehat{P}^{-h},\sigma_{k+1,n-2},\widehat{n-1})\mathcal{S}[\sigma_{k+1,n-2}|\widetilde{\sigma}_{k+1,n-2}]_{\widehat{p}_{n-1}}\widetilde{A}(\widehat{n-1},\widetilde{\sigma}_{k+1,n-2},-\widehat{P}^{-h})}{s_{k+1\ldots\widehat{n-1}}}\,, \tag{5.75}$$

where we used $s_{\widehat{1}2\ldots kn} = s_{k+1\ldots\widehat{n-1}}$ and already removed the vanishing mixed-helicity terms. The first part is a lower-point version of eq. (5.5), and the second part the regularized dual KLT form, *i.e.* we have

$$\sum_h \frac{M_{k+2}(\widehat{1},2,\ldots,k,n,\widehat{P}^h)M_{n-k}(k+1,\ldots,\widehat{n-1},-\widehat{P}^{-h})}{s_{12\ldots kn}}\,. \tag{5.76}$$

Once again we obtain the correct BCFW expression for all $s_{12\ldots kn}$ poles, and all poles related to these by a permutation of $2,3,\ldots,n-2$.

This covers all residues in eq. (5.38), and show that they indeed make up the full BCFW-expansion for a n-point gravity amplitude, and therefore $R_n = M_n$, see figure 5.2. Notice how the properties/relations from above sections played important roles; the BCFW method was the main tool for the whole proof, the BCJ-relations were needed not only for showing that all expressions in eq. (5.3), for different j-values, are equivalent, but also to argue for the vanishing of contributions from (A), and both the vanishing identities and regularized KLT-relations were important in identifying contributions from (B) with terms in a BCFW-expansion of M_n. We stress that no other crossing symmetry than what was already manifest has been used, so the identification with a gravity amplitude is also an indirect proof of the total crossing symmetry of the right-hand side of eq. (5.3) and (5.20).

Maybe the most important thing to take from this proof is, that it very explicitly illustrates how constrained scattering amplitudes are just from their general analytical properties. These constraints are so strong that they force perturbative gravity and gauge theories to be related through the KLT-relations, although a priori these theories seem completely unrelated.

5.6 Squaring of Numerators

The last point we want to address in this chapter is the consequence of using the BCJ-representation, reviewed in section 3.2, in combination with the KLT-relations. This results in an alternative way of looking at gravity as the square of gauge theories.

Using the four-point amplitudes from eq. (3.61) in the four-point KLT-relation in eq. (5.8) we see that

$$\begin{aligned} M_4(1,2,3,4) =& -s_{12}A_4(1,2,3,4)\widetilde{A}_4(1,2,4,3) \\ =& -s_{12}\left(\frac{n_s}{s_{12}}+\frac{n_t}{s_{14}}\right)\left(-\frac{\widetilde{n}_u}{s_{13}}-\frac{\widetilde{n}_s}{s_{12}}\right) \\ =& \frac{n_s\widetilde{n}_s}{s_{12}}+\frac{n_s\widetilde{n}_u}{s_{13}}+s_{12}\frac{n_t\widetilde{n}_u}{s_{14}s_{13}}+\frac{n_t\widetilde{n}_s}{s_{14}}\,, \end{aligned} \tag{5.77}$$

which, from $s_{12}=-s_{13}-s_{14}$ and the numerator Jacobi identity $n_s-n_u-n_t=0$, can be rewritten into

$$\begin{aligned} M_4(1,2,3,4) &= \frac{n_s\widetilde{n}_s}{s_{12}}+\frac{n_u\widetilde{n}_u}{s_{13}}+\frac{n_t\widetilde{n}_u}{s_{13}}-(s_{13}+s_{14})\frac{n_t\widetilde{n}_u}{s_{14}s_{13}}+\frac{n_t\widetilde{n}_s}{s_{14}} \\ &= \frac{n_s\widetilde{n}_s}{s_{12}}+\frac{n_u\widetilde{n}_u}{s_{13}}+\frac{n_t(\widetilde{n}_s-\widetilde{n}_u)}{s_{14}} \\ &= \frac{n_s\widetilde{n}_s}{s_{12}}+\frac{n_u\widetilde{n}_u}{s_{13}}+\frac{n_t\widetilde{n}_t}{s_{14}}\,. \end{aligned} \tag{5.78}$$

This is the the same form (up to coupling constants) as eq. (3.65) with the replacement $c_i \to \widetilde{n}_i$. The remarkable thing is that this feature generalizes to higher points [74], *i.e.* assuming we have found a BCJ-representation for the gauge-theory amplitude

$$\mathcal{A}_n = \sum_i \frac{c_i n_i}{(\prod_j s_j)_i}\,, \tag{5.79}$$

we simply replace the c_i's with another copy of gauge-theory numerators $\widetilde{n}_i$ (can be the same or different from n_i) to obtain the gravity amplitude

$$M_n = \sum_i \frac{\widetilde{n}_i n_i}{(\prod_j s_j)_i}\,. \tag{5.80}$$

At the level of BCJ-numerators the KLT-relations therefore correspond to the squaring of gauge-theory numerators.

The KLT-relations are not only valid as a mapping between Yang-Mills theory and Einstein gravity. As we will soon see, the same relations can be used to map between supersymmetric Yang-Mills and supersymmetric gravity theories. In such cases the product can be between gauge-theory amplitudes with different particle content. At the numerator level this means that n_i and $\widetilde{n}_i$ can be from different types of gauge-theory amplitudes and the squaring will result in the amplitude belonging to the gravity theory corresponding to the gauge-theory product. Before going into the mapping between supersymmetric gauge and gravity theories in the next chapter, let us for completeness briefly comment on representing the full gauge-theory amplitude in terms of KLT-relations.

Since eq. (5.79) and (5.80) take the same form and the rewriting of the gravity amplitude into eq. (5.80) only relies on the Jacobi identity, this immediately suggests an alternative way of writing the full gauge-theory amplitude [66]. Let us with $A_n^s(1,2,\ldots,n)$ denote the color-ordered amplitude calculated from the scalar theory with only cubic vertices given by the structure constants f^{abc}. The full color-dressed gluon amplitude can then be written in KLT-form as

$$\mathcal{A}_n = (-1)^{n+1}\sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A_n^s(n-1,n,\widetilde{\sigma}_{2,n-2},1)\mathcal{S}[\widetilde{\sigma}_{2,n-2}|\sigma_{2,n-2}]_{k_1}A_n(1,\sigma_{2,n-2},n-1,n)\,, \tag{5.81}$$

where A_n is just the usual color-ordered gluon amplitudes. These relations were also proven recursively in [67] along with the KK- and BCJ-relations for A_n^s.

Chapter 6

KLT-Relations in Supersymmetric Theories

In this chapter we examine the KLT-relations between supersymmetric Yang-Mills and gravity theories. We will use the superfield formalism reviewed in section 2.5 and investigate how products of super-Yang-Mills amplitudes with varying degree of supersymmetry maps to supersymmetric gravity amplitudes through KLT.

6.1 Maximally Supersymmetric KLT-Relations

It is well-known that the particle states of $\mathcal{N}_G = 8$ supergravity theory can be written in terms of tensor products between states of two $\mathcal{N} = 4$ super-Yang-Mills theories, see *e.g.* [51]. The 1×2 graviton states $h^{\pm}$ come from

$$(+1) \otimes (+1) \;\text{ and }\; (-1) \otimes (-1) \;,$$

the 8×2 gravitino states $\psi^{\pm}$ from

$$(+1/2)^4 \otimes (+1) \;,\; (+1) \otimes (+1/2)^4 \;\text{ and }\; (-1/2)^4 \otimes (-1) \;,\; (-1) \otimes (-1/2)^4,$$

the 28×2 vector states $v^{\pm}$ from

$$(0)^6 \otimes (+1) \,,\, (+1/2)^4 \otimes (+1/2)^4 \,,\, (+1) \otimes (0)^6 \;\text{ and }\; (0)^6 \otimes (-1) \,,\, (-1/2)^4 \otimes (-1/2)^4 \,,\, (-1) \otimes (0)^6 \,,$$

the 56×2 spin-$1/2$ fermions $\chi^{\pm}$ from

$$(-1/2)^4 \otimes (+1) \;,\; (0)^6 \otimes (+1/2)^4 \;,\; (+1/2)^4 \otimes (0)^6 \;,\; (+1) \otimes (-1/2)^4 \;,$$

$$(+1/2)^4 \otimes (-1) \;,\; (0)^6 \otimes (-1/2)^4 \;,\; (-1/2)^4 \otimes (0)^6 \;,\; (-1) \otimes (+1/2)^4 \;,$$

and finally the 70 scalars

$$(-1) \otimes (+1) \;,\; (-1/2)^4 \otimes (+1/2)^4 \;,\; (0)^4 \otimes (0)^4 \;,\; (+1/2)^4 \otimes (-1/2)^4 \;,\; (+1) \otimes (-1) \;.$$

The superscripts denote the degeneracy of states. In terms of the diamond diagrams, introduced in section 2.5, this is represented in figure 6.1.

At the amplitude level this (supergravity)$_{\mathcal{N}_G=8}$ = (super Yang-Mills)$_{\widetilde{\mathcal{N}}=4}\otimes$ (super Yang-Mills)$_{\mathcal{N}=4}$ "squaring" manifest itself as a supersymmetric version of the KLT-relations we considered in last chapter. Using a supersymmetric generalization of the BCFW recursion

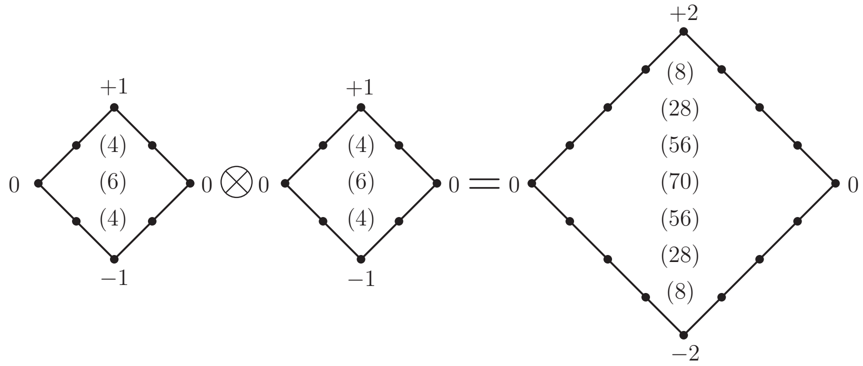


Figure 6.1: Diamond diagrams demonstrating the matching of states in (supergravity)$_{\mathcal{N}_G=8}$ = (super Yang-Mills)$_{\widetilde{\mathcal{N}}=4}\otimes$ (super Yang-Mills)$_{\mathcal{N}=4}$. The numbers inside the diamonds indicate the number of states on each line, and the number next to the dots indicate the helicities. Only the highest, lowest and zero helicities have been labeled explicitly.

relations, it was shown in [26] that the KLT-relations directly generalizes to superamplitudes between $\mathcal{N}_G = 8$ supergravity and two copies of $\mathcal{N} = 4$ super-Yang-Mills theory

$$\mathcal{M}_n^{\mathcal{N}_G=8} = (-1)^{n+1}\sum_{\gamma,\beta\in S_{n-3}} \widetilde{\mathcal{A}}_n^{\widetilde{\mathcal{N}}=4}(n-1,n,\gamma_{2,n-2},1)\mathcal{S}[\gamma_{2,n-2}|\beta_{2,n-2}]_{p_1}\mathcal{A}_n^{\mathcal{N}=4}(1,\beta_{2,n-2},n-1,n)\,. \tag{6.1}$$

When expanding out the right-hand side in terms of Grassmann variables η, we assign the $SU(4)_R$ R-indices $a = 1,2,3,4$ to the η_a variables in $\widetilde{\mathcal{A}}_n^{\widetilde{\mathcal{N}}=4}$ and the $SU(4)_R$ R-indices $b = 5,6,7,8$ to the η_b variables in $\mathcal{A}_n^{\mathcal{N}=4}$. The product gives strings of η_a and η_b variables with combined indices $1,2,\dots,8$ whose coefficients are KLT-products between component helicity amplitudes. By identifying these strings of Grassmann variables with the corresponding monomials of η_A, $A = 1,2,\dots,8$, in the expansion of the gravity superamplitude on the left-hand side, we get the full set of KLT-relations between all component helicity amplitudes.

As an example, consider the KLT-product between the two four-point amplitudes $\widetilde{A}_4(g_3^-,g_4^+,f_2^-,f_1^-)$ and $A_4(f_1^+,f_2^-,g_3^-,g_4^+)$, with the following η-monomials

$$\begin{aligned}\widetilde{A}_4(g_3^-,g_4^+,f_2^-,f_1^+) &\sim (\eta_{31}\eta_{32}\eta_{33}\eta_{34})(\eta_{22}\eta_{23}\eta_{24})(\eta_{11})\,,\\ A_4(f_1^+,f_2^-,g_3^-,g_4^+) &\sim (\eta_{18})(\eta_{25}\eta_{26}\eta_{27})(\eta_{35}\eta_{36}\eta_{37}\eta_{38})\,.\end{aligned} \tag{6.2}$$

The combined η-string is

$$(\eta_{11}\eta_{18})(\eta_{22}\eta_{23}\eta_{24}\eta_{25}\eta_{26}\eta_{27})(\eta_{31}\eta_{32}\eta_{33}\eta_{34}\eta_{35}\eta_{36}\eta_{37}\eta_{38})\,. \tag{6.3}$$

On the left-hand side of eq. (6.1) this corresponds to a $M_4(v_1^+,v_2^-,h_3^-,h_4^+)$ amplitude, involving graviphotons and gravitons, *i.e.*

$$M_4(v_1^+,v_2^-,h_3^-,h_4^+) = s_{12}\widetilde{A}_4(g_3^-,g_4^+,f_2^-,f_1^+)A_4(f_1^+,f_2^-,g_3^-,g_4^+)\,. \tag{6.4}$$

For notational simplicity we have omitted the R-symmetry indices on the particle states in the amplitudes. They can be read off from the η-strings, see eq. (2.75) and (2.91).

Eq. (6.1) does not only give all the KLT-relations, but also all the vanishing identities, like those introduced in section 5.4. Recall that $\widetilde{\mathcal{A}}_n^{\widetilde{\mathcal{N}}=4}$ is invariant under $SU(4)_R$. Each monomial in the η-expansion, for all non-vanishing component helicity amplitudes, must therefore have each η_a ($a = 1,2,3,4$) to the same power k. Likewise $\mathcal{A}_n^{\mathcal{N}=4}$ is also invariant under $SU(4)_R$ and has the same power k' for each η_b ($b = 5,6,7,8$) in its monomials. In the product, on the right-hand side, there will be terms where the power k is *not* equal to k'. Such expressions would violate $SU(8)_R$ R-symmetry, which gravity superamplitude possess, and we therefore conclude that the coefficients to such $SU(8)_R$ violating η-strings must vanish. These coefficients make up all the vanishing identities. For example, a KLT-product like

$$\sum_{\gamma,\beta\in S_{n-3}} \widetilde{A}_n^{MHV}(n-1,n,\gamma_{2,n-2},1)\mathcal{S}[\gamma_{2,n-2}|\beta_{2,n-2}]_{p_1} A_n^{NMHV}(1,\beta_{2,n-2},n-1,n)\,, \quad (6.5)$$

is the coefficient to an η-string with power 2 for the indices $1,\ldots,4$ and power 3 for $5,\ldots,8$, which violates $SU(8)_R$ symmetry and therefore vanish altogether.

6.2 KLT-Relations in Less Supersymmetric Theories

In section 2.5 we saw how to get superamplitudes for theories with less than maximal supersymmetry by truncating or integrating out η-variables in superamplitudes for maximally supersymmetric theories. Following the analysis in ref. [28], we can use this in the KLT-product to investigate how different super-Yang-Mills theories map to supergravity through the KLT-relations.

Starting from the KLT-product between two super-Yang-Mills superamplitudes of arbitrary supersymmetry $\widetilde{\mathcal{N}} \leq 4$ and $\mathcal{N} \leq 4$, see eq. (2.97), we get (omitting the overall sign-factor)

$$\begin{aligned}
&\sum_{\gamma,\beta\in S_{n-3}} \widetilde{\mathcal{A}}^{\widetilde{\mathcal{N}}\leq 4}_{n,\tilde{i}_1,\ldots\tilde{i}_{\tilde{m}}}(n-1,n,\gamma_{2,n-2},1)\mathcal{S}[\gamma_{2,n-2}|\beta_{2,n-2}]_{p_1}\mathcal{A}^{\mathcal{N}\leq 4}_{n,i_1\ldots i_m}(1,\beta_{2,n-2},n-1,n)\\
&= \sum_{\gamma,\beta\in S_{n-3}}\left[\int \prod_{\tilde{a}_1=\widetilde{\mathcal{N}}+1}^{4} d\eta_{\tilde{i}_1,\tilde{a}_1}\cdots \prod_{\tilde{a}_{\tilde{m}}=\widetilde{\mathcal{N}}+1}^{4} d\eta_{\tilde{i}_{\tilde{m}},\tilde{a}_{\tilde{m}}}\widetilde{\mathcal{A}}_n^{\widetilde{\mathcal{N}}=4}(n-1,n,\gamma_{2,n-2},1)\right]_{\eta_{\widetilde{\mathcal{N}}+1},\ldots,\eta_4\to 0}\\
&\times\mathcal{S}[\gamma_{2,n-2}|\beta_{2,n-2}]_{p_1}\times\left[\int\prod_{a_1=\mathcal{N}+5}^{8} d\eta_{i_1,a_1}\cdots\prod_{a_m=\mathcal{N}+5}^{8} d\eta_{i_m,a_m}\mathcal{A}_n^{\mathcal{N}=4}(1,\beta_{2,n-2},n-1,n)\right]_{\eta_{\mathcal{N}+5},\ldots,\eta_8\to 0}\\
&=\left[\int \prod_{\tilde{a}_1=\widetilde{\mathcal{N}}+1}^{4} d\eta_{\tilde{i}_1,\tilde{a}_1}\cdots \prod_{\tilde{a}_{\tilde{m}}=\widetilde{\mathcal{N}}+1}^{4} d\eta_{\tilde{i}_{\tilde{m}},\tilde{a}_{\tilde{m}}}\prod_{a_1=\mathcal{N}+5}^{8} d\eta_{i_1,a_1}\cdots\prod_{a_m=\mathcal{N}+5}^{8} d\eta_{i_m,a_m}\right.\\
&\left.\times\sum_{\gamma,\beta\in S_{n-3}}\widetilde{\mathcal{A}}_n^{\widetilde{\mathcal{N}}=4}(n-1,n,\gamma_{2,n-2},1)\mathcal{S}[\gamma_{2,n-2}|\beta_{2,n-2}]_{p_1}\mathcal{A}_n^{\mathcal{N}=4}(1,\beta_{2,n-2},n-1,n)\right]_{\substack{\eta_{\widetilde{\mathcal{N}}+1},\ldots,\eta_4\to 0\\ \eta_{\mathcal{N}+5},\ldots,\eta_8\to 0}}\\
&=\left[\int \prod_{\tilde{a}_1=\widetilde{\mathcal{N}}+1}^{4} d\eta_{\tilde{i}_1,\tilde{a}_1}\cdots \prod_{\tilde{a}_{\tilde{m}}=\widetilde{\mathcal{N}}+1}^{4} d\eta_{\tilde{i}_{\tilde{m}},\tilde{a}_{\tilde{m}}}\right.\\
&\qquad\left.\times\prod_{a_1=\mathcal{N}+5}^{8} d\eta_{i_1,a_1}\cdots\prod_{a_m=\mathcal{N}+5}^{8} d\eta_{i_m,a_m}\mathcal{M}_n^{\mathcal{N}_G=8}(\Phi_1,\Phi_2,\ldots,\Phi_n)\right]_{\substack{\eta_{\widetilde{\mathcal{N}}+1},\ldots,\eta_4\to 0\\ \eta_{\mathcal{N}+5},\ldots,\eta_8\to 0}}\\
&\equiv \mathcal{M}^{\mathcal{N}_G\leq 8}_{n,(\tilde{i}_1,\ldots,\tilde{i}_{\tilde{m}});(i_1,\ldots,i_m)}\,,
\end{aligned}\quad (6.6)$$

where the subscripts $(\tilde{i}_1, \ldots, \tilde{i}_{\widetilde{m}})$ and $(i_1, \ldots, i_m)$ label the external legs given by $\widetilde{\Psi}$ and Ψ fields, respectively (here $\widetilde{\Phi}$ and $\widetilde{\Psi}$ is just shorthand for $\Phi^{\widetilde{\mathcal{N}}}$ and $\Psi^{\widetilde{\mathcal{N}}}$, respectively). $\widetilde{m} \leq n$ and $m \leq n$. In the second to last step, we used the $\mathcal{N}_G = 8$ super KLT-relation. $\mathcal{M}_n^{\mathcal{N}_G \leq 8}$ is the superamplitude for the resulting $\mathcal{N}_G \leq 8$ supergravity theory, obtained from $\mathcal{N}_G = 8$ by truncating and integrating out η's as dictated above. In terms of the superfields this give the following four possibilities for each external leg k in the supergravity superamplitude:

- $(\widetilde{\Phi}, \Phi)$: if $k \notin (\tilde{i}_1, \ldots, \tilde{i}_{\widetilde{m}})$ and $k \notin (i_1, \ldots, i_m)$, we set all $\eta_{k,\widetilde{\mathcal{N}}+1}, \ldots, \eta_{k,4}$ and $\eta_{k,\mathcal{N}+5}, \ldots, \eta_{k,8}$ to zero, and the resulting superfield is

$$\Phi_k^{\mathcal{N}_G = \widetilde{\mathcal{N}} + \mathcal{N}} = \Phi_k^{\mathcal{N}_G = 8}|_{\eta_{k,\widetilde{\mathcal{N}}+1}, \ldots, \eta_{k,4}; \eta_{k,\mathcal{N}+5}, \ldots, \eta_{k,8} \to 0} \ . \tag{6.7}$$

- $(\widetilde{\Psi}, \Psi)$: if $k \in (\tilde{i}_1, \ldots, \tilde{i}_{\widetilde{m}})$ and $k \in (i_1, \ldots, i_m)$, we get

$$\Psi_k^{\mathcal{N}_G = \widetilde{\mathcal{N}} + \mathcal{N}} = \int \prod_{a=\widetilde{\mathcal{N}}+1}^{4} d\eta_{k,a} \prod_{b=\mathcal{N}+5}^{8} d\eta_{k,b} \Phi_k^{\mathcal{N}_G = 8} \ . \tag{6.8}$$

  These two superfields combine to form a full $SU(\mathcal{N}_G)$ supergravity multiplet.

- $(\widetilde{\Psi}, \Phi)$: if $k \in (\tilde{i}_1, \ldots, \tilde{i}_{\widetilde{m}})$ and $k \notin (i_1, \ldots, i_m)$, we have

$$\Theta_k^{\mathcal{N}_G = \widetilde{\mathcal{N}} + \mathcal{N}} = \int \prod_{a=\widetilde{\mathcal{N}}+1}^{4} d\eta_{k,a} \Phi_k^{\mathcal{N}_G = 8}|_{\eta_{k,\mathcal{N}+5}, \ldots, \eta_{k,8} \to 0} \ . \tag{6.9}$$

- $(\widetilde{\Phi}, \Psi)$: if $k \notin (\tilde{i}_1, \ldots, \tilde{i}_{\widetilde{m}})$ and $k \in (i_1, \ldots, i_m)$, we have

$$\Gamma_k^{\mathcal{N}_G = \widetilde{\mathcal{N}} + \mathcal{N}} = \int \prod_{b=\mathcal{N}+5}^{8} d\eta_{k,b} \Phi_k^{\mathcal{N}_G = 8}|_{\eta_{k,\widetilde{\mathcal{N}}+1}, \ldots, \eta_{k,4} \to 0} \ . \tag{6.10}$$

  The latter two superfields combine to form a $SU(\mathcal{N}_G)$ matter supermultiplet, if $\widetilde{\mathcal{N}} < 3$ and $\mathcal{N} < 3$.

Thus, in general, we need four superfields to encode the content of the gravity theory resulting from a product between two minimal super-Yang-Mills theories of arbitrary degree of supersymmetry. Going through all possible cases one finds the results summarised in table 6.1. Let us go through some explicit examples to show how the table is obtained.

Examples

$(\widetilde{\mathcal{N}} = 4) \otimes (\mathcal{N} = 2)$

The particle content of the product between $\widetilde{\mathcal{N}} = 4$ and $\mathcal{N} = 2$ super Yang-Mills is described by the two superfields

$$\begin{aligned} \Phi^{\mathcal{N}_G = 6} &= \Phi^{\mathcal{N}_G = 8}|_{\eta_7, \eta_8 \to 0} \\ &= h_+ + \sum_{i=1,2,3,4,5,6} \eta_i \psi_+^i + \sum_{i<j=1,2,3,4,5,6} \eta_i \eta_j v_+^{ij} + \sum_{i<j<k=1,2,3,4,5,6} \eta_i \eta_j \eta_k \chi_+^{ijk} \\ &\quad + \sum_{i<j<k<l=1,2,3,4,5,6} \eta_i \eta_j \eta_k \eta_l \phi^{ijkl} + \sum_{i<j<k<l<m=1,2,3,4,5,6} \eta_i \eta_j \eta_k \eta_l \eta_m \chi_-^{ijklm} \\ &\quad + \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 v_-^{123456} \ , \end{aligned} \tag{6.11}$$

$\widetilde{\mathcal{N}} \otimes \mathcal{N}$	Description
$4 \otimes 4$	Maximal $\mathcal{N}_G = 8$ Supergravity
$4 \otimes 3$	Maximal $\mathcal{N}_G = 8$ Supergravity
$4 \otimes 2$	Minimal $\mathcal{N}_G = 6$ Supergravity with $SU(6)$ supergravity multiplet
$3 \otimes 3$	Maximal $\mathcal{N}_G = 8$ Supergravity
$4 \otimes 1$	Minimal $\mathcal{N}_G = 5$ Supergravity with $SU(5)$ supergravity multiplet
$3 \otimes 2$	Minimal $\mathcal{N}_G = 6$ Supergravity with $SU(6)$ supergravity multiplet
$4 \otimes 0$	Minimal $\mathcal{N}_G = 4$ Supergravity with $SU(4)$ supergravity multiplet
$3 \otimes 1$	Minimal $\mathcal{N}_G = 5$ Supergravity with $SU(5)$ supergravity multiplet
$2 \otimes 2$	$\mathcal{N}_G = 4$ Supergravity multiplet coupled to vector multiplet
$3 \otimes 0$	Minimal $\mathcal{N}_G = 4$ Supergravity with $SU(4)$ supergravity multiplet
$2 \otimes 1$	$\mathcal{N}_G = 3$ Supergravity multiplet coupled to vector multiplet
$2 \otimes 0$	$\mathcal{N}_G = 2$ Supergravity multiplet coupled to vector multiplet
$1 \otimes 1$	$\mathcal{N}_G = 2$ Supergravity multiplet coupled to hypermultiplet
$1 \otimes 0$	$\mathcal{N}_G = 1$ Supergravity multiplet coupled to chiral multiplet
$0 \otimes 0$	Einstein gravity coupled to two scalars

Table 6.1: All possible supergravity theories that can be constructed from KLT-products between minimal super-Yang-Mills theories with varying degree of supersymmetry.

and

$$\begin{aligned}
\Psi^{\mathcal{N}_G=6} &= \int d\eta_7 d\eta_8 \Phi^{\mathcal{N}_G=8} \\
&= -v_+^{(78)} - \sum_{i=1,2,3,4,5,6} \eta_i \chi_+^{i(78)} - \sum_{i<j=1,2,3,4,5,6} \eta_i\eta_j \phi^{ij(78)} \\
&\quad - \sum_{i<j<k=1,2,3,4,5,6} \eta_i\eta_j\eta_k \chi_-^{ijk(78)} - \sum_{i<j<k<l=1,2,3,4,5,6} \eta_i\eta_j\eta_k\eta_l v_-^{ijkl(78)} \\
&\quad - \sum_{i<j<k<l<m=1,2,3,4,5,6} \eta_i\eta_j\eta_k\eta_l\eta_m \psi_-^{ijklm(78)} - \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6 h_-^{123456(78)} ,
\end{aligned} \tag{6.12}$$

represented by the diamond diagrams in figure 6.2. This is exactly the content of minimal $\mathcal{N}_G = 6$ supergravity with its two supermultiplets $(+2, +{}^3\!/{}_2{}^6, +1^{15}, +{}^1\!/{}_2{}^{20}, 0^{15}, -{}^1\!/{}_2{}^6, -1)$ and $(+1, +{}^1\!/{}_2{}^6, 0^{15}, -{}^1\!/{}_2{}^{20}, -1^{15}, -{}^3\!/{}_2{}^6, -2)$. The corresponding KLT-relations are given by

$$\begin{aligned}
&\mathcal{M}_n^{\mathcal{N}_G=6}(\Phi_{i_1,\ldots,i_{m_1}}^{\mathcal{N}_G=6}, \Psi_{j_1,\ldots,j_{m_2}}^{\mathcal{N}_G=6}) = \\
&\qquad \sum_{\gamma,\beta\in S_{n-3}} \widetilde{\mathcal{A}}_n^{\widetilde{\mathcal{N}}=4}(\Phi_{1,\ldots,n}^{\widetilde{\mathcal{N}}=4}) \times \mathcal{S}[\gamma|\beta]_{p_1} \times \mathcal{A}_n^{\mathcal{N}=2}(\Phi_{i_1,\ldots,i_{m_1}}^{\mathcal{N}=2}, \Psi_{j_1,\ldots,j_{m_2}}^{\mathcal{N}=2}) ,
\end{aligned} \tag{6.13}$$

where the indices $(i_1, \ldots, i_{m_1})$ and $(j_1, \ldots, j_{m_2})$ denote the legs of the corresponding superfields and $m_1+m_2 = n$. The ordering of the superfields in the super-Yang-Mills amplitudes are of course still like in eq. (6.1), which has been suppressed here for simplicity.

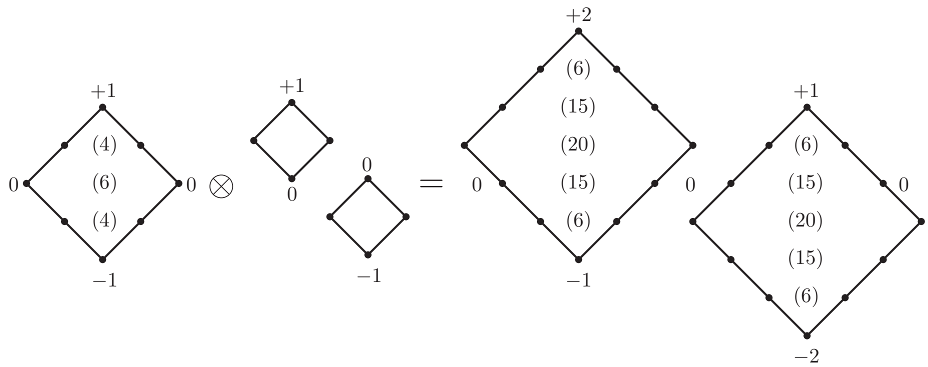


Figure 6.2: Diamond diagram for $(\text{supergravity})_{\mathcal{N}_G=6} = (\text{super Yang-Mills})_{\widetilde{\mathcal{N}}=4} \otimes (\text{super Yang-Mills})_{\mathcal{N}=2}$. The two diamonds at the right-hand side represent the $\Phi^{\mathcal{N}_G=6}$ and $\Psi^{\mathcal{N}_G=6}$ superfields, respectively. There are two hidden indices (78) for the Ψ field.

$(\widetilde{\mathcal{N}} = 2) \otimes (\mathcal{N} = 2)$

The particle content of the product between $\widetilde{\mathcal{N}} = 2$ and $\mathcal{N} = 2$ super Yang-Mills takes four superfields to describe

$$\Phi^{\mathcal{N}_G=4} = \Phi^{\mathcal{N}_G=8}|_{\eta_3,\eta_4,\eta_7,\eta_8\to 0} = h_+ + \sum_{i=1,2,5,6} \eta_i \psi_+^i + \sum_{i<j=1,2,5,6} \eta_i\eta_j v_+^{ij} + \sum_{i<j<k=1,2,5,6} \eta_i\eta_j\eta_k \chi_+^{ijk} + \eta_1\eta_2\eta_5\eta_6 \phi^{1256}\,, \tag{6.14}$$

its CPT-conjugate

$$\Psi^{\mathcal{N}_G=4} = \int d\eta_3 d\eta_4 d\eta_7 d\eta_8 \Phi^{\mathcal{N}_G=8} = \phi^{(3478)} + \sum_{i=1,2,5,6} \eta_i \chi_-^{i(3478)} + \sum_{i<j=1,2,5,6} \eta_i\eta_j v_-^{ij(3478)} + \sum_{i<j<k=1,2,5,6} \eta_i\eta_j\eta_k \psi_-^{ijk(3478)} + \eta_1\eta_2\eta_5\eta_6 h_-^{1256(3478)}\,, \tag{6.15}$$

and

$$\Theta_{vector}^{\mathcal{N}_G=4} \equiv \int d\eta_3 d\eta_4 \Phi^{\mathcal{N}_G=8}|_{\eta_7,\eta_8\to 0} = -\, v_+^{(34)} - \sum_{i=1,2,5,6} \eta_i \chi_+^{i(34)} - \sum_{i<j=1,2,5,6} \eta_i\eta_j \phi^{ij(34)} - \sum_{i<j<k=1,2,5,6} \eta_i\eta_j\eta_k \chi_-^{ijk(34)} - \eta_1\eta_2\eta_5\eta_6 v_-^{1256(34)}\,, \tag{6.16}$$

and its CPT-conjugate

$$\Gamma_{vector}^{\mathcal{N}_G=4} \equiv \int d\eta_7 d\eta_8 \Phi^{\mathcal{N}_G=8}|_{\eta_3,\eta_4\to 0} = -\, v_+^{(78)} - \sum_{i=1,2,5,6} \eta_i \chi_+^{i(78)} - \sum_{i<j=1,2,5,6} \eta_i\eta_j \phi^{ij(78)} - \sum_{i<j<k=1,2,5,6} \eta_i\eta_j\eta_k \chi_-^{ijk(78)} - \eta_1\eta_2\eta_5\eta_6 v_-^{1256(78)}\,. \tag{6.17}$$

These are represented by diamond diagrams in figure 6.3.

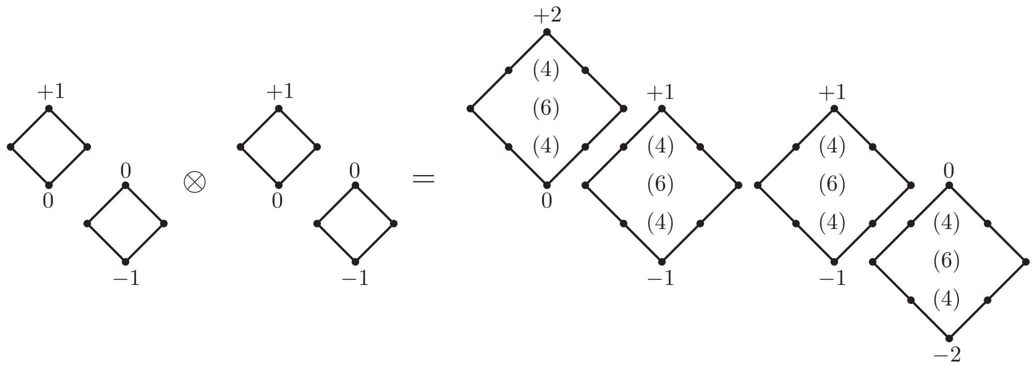


Figure 6.3: Diamond diagram for (supergravity)$_{\mathcal{N}_G=4}$ = (super Yang-Mills)$_{\widetilde{\mathcal{N}}=2}\otimes$ (super Yang-Mills)$_{\mathcal{N}=2}$. The four diamonds at the right-hand side represent the $\Phi^{\mathcal{N}_G=4}$, $\Theta^{\mathcal{N}_G=4}_{vector}$, $\Gamma^{\mathcal{N}_G=4}_{vector}$ and $\Psi^{\mathcal{N}_G=4}$ superfields. Θ and Γ correspond to CPT self-conjugate vector multiplets, but they have different sets of $SU(4)_R$ indices. The hidden indices are (34) for Θ, (78) for Γ and (3478) for Ψ.

Besides the minimal $\mathcal{N}_G = 4$ supergravity multiplet this theory contains two additional superfields (or diamonds). These represent a matter (vector) multiplet, consisting of 2 vector fields with helicity ± 1, 8 fermion fields of helicity $\pm 1/2$ and 12 scalars. Hence the resulting theory from the $(\widetilde{\mathcal{N}} = 2) \otimes (\mathcal{N} = 2)$ product is minimal $\mathcal{N}_G = 4$ supergravity coupled to these matter fields.

The corresponding KLT-relations can be written as

$$\begin{aligned}\mathcal{M}_n^{\mathcal{N}_G=4}(\Phi^{\mathcal{N}_G=4}_{i_1,\ldots,i_{m_1}}, \Psi^{\mathcal{N}_G=4}_{j_1,\ldots,j_{m_2}}, \Theta^{\mathcal{N}_G=4}_{k_1,\ldots,k_{m_3}}, \Gamma^{\mathcal{N}_G=4}_{l_1,\ldots,l_{m_4}}) = \\ \sum_{\gamma,\beta\in S_{n-3}} \widetilde{\mathcal{A}}_n^{\widetilde{\mathcal{N}}=2}(\Phi^{\widetilde{\mathcal{N}}=2}_{i_1,\ldots,i_{m_1},l_1,\ldots,l_{m_4}}, \Psi^{\widetilde{\mathcal{N}}=2}_{j_1,\ldots,j_{m_2},k_1,\ldots,k_{m_3}}) \\ \times \mathcal{S}[\gamma|\beta]_{p_1} \times \mathcal{A}_n^{\mathcal{N}=2}(\Phi^{\mathcal{N}=2}_{i_1,\ldots,i_{m_1},k_1,\ldots,k_{m_3}}, \Psi^{\mathcal{N}=2}_{j_1,\ldots,j_{m_2},l_1,\ldots,l_{m_4}}) \,, \qquad (6.18)\end{aligned}$$

where again $(i_1, \ldots, i_{m_1}), (j_1, \ldots, j_{m_2}), (k_1, \ldots, k_{m_3})$ and $(l_1, \ldots, l_{m_4})$ label legs of the corresponding superfields and $m_1 + m_2 + m_3 + m_4 = n$.

As a final example let us see how the $(\widetilde{\mathcal{N}} = 0) \otimes (\mathcal{N} = 0)$ case fits into this formalism.

$(\widetilde{\mathcal{N}} = 0) \otimes (\mathcal{N} = 0)$

The particle content of the product between two $\mathcal{N} = 0$ "super"-Yang-Mills theories is also described in terms of four "superfields"

$$\Phi^{\mathcal{N}_G=0} = \Phi^{\mathcal{N}_G=8}|_{\eta_1,\ldots,\eta_8\to 0} = h_+ \,, \qquad (6.19)$$

$$\Psi^{\mathcal{N}_G=0} = \int \prod_{A=1}^{8} d\eta_A \Phi^{\mathcal{N}_G=8} = h_-^{(12345678)} \,, \qquad (6.20)$$

and the two scalars

$$\Theta^{\mathcal{N}_G=0} = \int \prod_{a=1}^{4} d\eta_a \Phi^{\mathcal{N}_G=8}|_{\eta_5,\ldots,\eta_8\to 0} = \phi^{(1234)} \,, \qquad (6.21)$$

$$\Gamma^{\mathcal{N}_G=0} = \int \prod_{a=5}^{8} d\eta_a \Phi^{\mathcal{N}_G=8}|_{\eta_1,\ldots,\eta_4\to 0} = \phi^{(5678)} \,. \qquad (6.22)$$

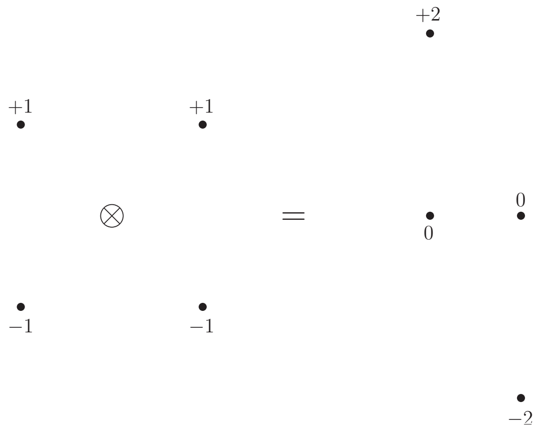


Figure 6.4: “Diamond” diagram for $(\text{gravity})_{\mathcal{N}_G=0} = (\text{Yang-Mills})_{\widetilde{\mathcal{N}}=0} \otimes$ $(\text{Yang-Mills})_{\mathcal{N}=0}$. This is just the classic case of gravity as the square of two Yang-Mills theories. The origin of the two additional scalars is clear in the language of hidden indices.

The “diamond” diagrams are given in figure 6.4.

This is nothing but Einstein gravity coupled to two scalars, which can be combined into an axion and a dilaton field. This theory can be obtained from the low-energy limit of string theory which was used in explaining the vanishing identities in the absent of supersymmetry [73].

6.3 Linear Symmetry Groups

We saw above how to reduce to gravity theories with less supersymmetry from truncation/integration of η-variables in the maximally supersymmetric case. However, we also argued that it was the violation of $SU(8)_R$ R-symmetry that explained the vanishing identities. A natural question arises; what does the truncation/integration imply for the linear symmetry group in the less supersymmetric theories? And in particular in connection with the vanishing identities. How does one explain the vanishing identities from the point of view of less supersymmetry? This will be the subject of this section.

Recall that an arbitrary (infinitesimal) $SU(8)_R$ transformation can be written as

$$e^{i\epsilon T} \simeq 1 + i\epsilon T \,, \tag{6.23}$$

where T is an element in the Lie-algebra of $SU(8)_R$, *i.e.* it is an 8×8 traceless Hermitian matrix, and $\epsilon > 0$ is some small number.

In general when we want to consider the effect of a matrix B (*e.g.* a sub-matrix embedded in T) acting on a (sub)set of adjacent R-symmetry indices, say $(x_1, x_2, \ldots, x_k)$, we are actually, because of the antisymmetry of these indices, looking at the effect of the matrix on $e_{x_1} \wedge e_{x_2} \wedge \cdots \wedge e_{x_k}$. Here $e_{x_i} = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ is the basis vector representing η_{x_i}, with 1 in the ith position. We have that

$$Be_{x_i} = \sum_{j=1}^{k} B_{ij} e_{x_j} \;, \tag{6.24}$$

and therefore

$$
\begin{aligned}
B(e_{x_1} \wedge e_{x_2} \wedge \cdots \wedge e_{x_k}) &= \sum_{i=1}^{k} e_{x_1} \wedge \cdots \wedge (Be_{x_i}) \wedge \cdots \wedge e_{x_k} \\
&= \sum_{i=1}^{k} e_{x_1} \wedge \cdots \wedge (\sum_{j=1}^{k} B_{ij} e_{x_j}) \wedge \cdots \wedge e_{x_k} \\
&= \sum_{i=1}^{k} e_{x_1} \wedge \cdots \wedge (B_{ii} e_{x_i}) \wedge \cdots \wedge e_{x_k} \\
&= (\sum_{i=1}^{k} B_{ii})(e_{x_1} \wedge e_{x_2} \wedge \cdots \wedge e_{x_k}) \,. \qquad (6.25)
\end{aligned}
$$

In the first line we used that we are looking at infinitesimal transformations and in the third line we used the antisymmetry of the $\wedge$ product. We see that only the trace of matrix B_{ij} is important. Note that this is only true when the indices are adjacent, such that they can be expressed as $(e_{x_1} \wedge e_{x_2} \wedge \cdots \wedge e_{x_k})$. If we denote $\beta = Tr(B_{ij})$, we can then assign a charge of β to the collection of indices $(x_1, x_2, \ldots, x_k)$.

Minimal $\mathcal{N}_G = 4, 5, 6$ Supergravity

Let us start by examining the linear symmetry groups for the minimal supergravity theories that could be obtained from products between minimal super-Yang-Mills theories. These were $\mathcal{N}_G = 4, 5, 6$ supergravity. We will make the analysis in the setting of minimal $\mathcal{N}_G = 6$ supergravity. As we saw above, this theory, with $SU(6)_R$ indices $1, 2, \ldots, 6$, can be obtained from $\mathcal{N}_G = 8$ supergravity by a truncation/integration of index 7 and 8. These are the hidden indices we denoted in parentheses in eq. (6.12). The traceless Hermitian $SU(6)_R$-generators $T_{6\times 6}$ are all embedded in the Lie-algebra of $SU(8)_R$ as

$$
\begin{pmatrix} T_{6\times 6} & \\ & C_{2\times 2} \end{pmatrix} , \qquad (6.26)
$$

where $C_{2\times 2}$ is some traceless Hermitian 2×2 matrix. However, the Lie-algebra of $SU(8)_R$ also contains matrices of the form

$$
\begin{pmatrix} \alpha I_{6\times 6} & \\ & B_{2\times 2} \end{pmatrix} , \qquad (6.27)
$$

where $I_{6\times 6}$ is the 6×6 identity matrix. This corresponds to a $U(1)$ symmetry assigning a charge α to each R-index $1, 2, \ldots, 6$. Since only the trace part of B will be involved when acting on the combined (78) indices, we can assign a charge $\beta \equiv Tr(B)$ to (78), and because of the traceless condition on $SU(8)$ generators we have the constraint $\beta = -6\alpha$.

We can then write down the total charge for each particle state. For the Φ field we find

Helicity	KLT Products	Charge
$+2$	$(+1) \otimes (+1)$	0
$+\frac{3}{2}$	$(+\frac{1}{2}^{a_1}) \otimes (+1)$, $(+1) \otimes (+\frac{1}{2}^{b_1})$	α
$+1$	$(0^{a_1 a_2}) \otimes (+1)$, $(+\frac{1}{2}^{a_1}) \otimes (+\frac{1}{2}^{b_1})$, $(+1) \otimes (0^{56})$	2α
$+\frac{1}{2}$	$(-\frac{1}{2}^{a_1 a_2 a_3}) \otimes (+1)$, $(0^{a_1 a_2}) \otimes (+\frac{1}{2}^{b_1})$, $(+\frac{1}{2}^{a_1}) \otimes (0^{56})$	3α
0	$(-1^{1234}) \otimes (+1)$, $(-\frac{1}{2}^{a_1 a_2 a_3}) \otimes (+\frac{1}{2}^{b_1})$, $(0^{a_1 a_2}) \otimes (0^{56})$	4α
$-\frac{1}{2}$	$(-1^{1234}) \otimes (+\frac{1}{2}^{b_1})$, $(-\frac{1}{2}^{a_1 a_2 a_3}) \otimes (0^{56})$	5α
-1	$(-1^{1234}) \otimes (0^{56})$	6α

where $a_i = 1,2,3,4$, and $b_1 = 5,6$. For the Ψ superfield we get

Helicity	KLT Products	Charge
$+1$	$(+1)\otimes(0^{(78)})$	-6α
$+\frac{1}{2}$	$(+1)\otimes(-\frac{1}{2}^{b_1(78)})$, $(+\frac{1}{2}^{a_1})\otimes(0^{(78)})$	-5α
0	$(+1)\otimes(-1^{56(78)})$, $(+\frac{1}{2}^{a_1})\otimes(-\frac{1}{2}^{b_1(78)})$, $(0^{a_1a_2})\otimes(0^{(78)})$	-4α
$-\frac{1}{2}$	$(+\frac{1}{2}^{a_1})\otimes(-1^{56(78)})$, $(0^{a_1a_2})\otimes(-\frac{1}{2}^{b_1(78)})$, $(-\frac{1}{2}^{a_1a_2a_3})\otimes(0^{(78)})$	-3α
-1	$(0^{a_1a_2})\otimes(-1^{56(78)})$, $(-\frac{1}{2}^{a_1a_2a_3})\otimes(-\frac{1}{2}^{b_1(78)})$, $(-1^{1234})\otimes(0^{(78)})$	-2α
$-\frac{3}{2}$	$(-\frac{1}{2}^{a_1a_2a_3})\otimes(-1^{56(78)})$, $(-1^{1234})\otimes(-\frac{1}{2}^{b_1(78)})$	$-\alpha$
-2	$(-1^{1234})\otimes(-1^{56(78)})$	0

Here we have written out the gravity states as they appear from the product between $\widetilde{\mathcal{N}}=4$ and $\mathcal{N}=2$ super-Yang-Mills theory. This makes it easier to write up the corresponding KLT-relation from a gravity amplitude.

We conclude that besides the $SU(6)_R$ symmetry group, $\mathcal{N}_G = 6$ supergravity also contains a non-trivial $U(1)_R$ group. The gravity amplitudes will of course still vanish if they violate $SU(6)_R$ symmetry, but from a $\mathcal{N}_G = 6$ supergravity point of view, the vanishing of, for instance,

$$M_5^{\mathcal{N}_G=6}(h_-^{123456(78)}, h_-^{123456(78)}, v_-^{123456}, h_+, h_+)\,, \tag{6.28}$$

which is $SU(6)_R$ invariant, is due to breaking of $U(1)_R$-charge conservation (the total charge is 6α.)

In a similar manner we find an additional $U(1)_R$ group for the minimal $\mathcal{N}_G = 5$ and $\mathcal{N}_G = 4$ supergravity theories, extending their linear symmetry group from $SU(\mathcal{N}_G)_R$ to $U(\mathcal{N}_G)_R = SU(\mathcal{N}_G)_R \otimes U(1)_R$.

Minimal $\mathcal{N}_G = 0,1,2,3,4$ Supergravity Coupled to Matter

We now consider the cases where the product between two super-Yang-Mills theories resulted in a supergravity theory coupled to matter multiplets. To make the discussion more concrete we use the example of $(\widetilde{\mathcal{N}}=2)\otimes(\mathcal{N}=2)$ from above.

The hidden variables are (34) and (78), so the $SU(4)_R$ indices for this theory are $1,2,5,6$. The $T_{4\times4}$ generators are embedded into the Lie-algebra of $SU(8)_R$ as

$$\begin{pmatrix} T^1_{2\times2} & & T^2_{2\times2} & \\ & B_{2\times2} & & \\ T^3_{2\times2} & & T^4_{2\times2} & \\ & & & C_{2\times2} \end{pmatrix}, \tag{6.29}$$

where

$$T_{4\times4} = \begin{pmatrix} T^1_{2\times2} & T^2_{2\times2} \\ T^3_{2\times2} & T^4_{2\times2} \end{pmatrix}. \tag{6.30}$$

The Lie-algebra of $SU(8)_R$ also contains elements like

$$\begin{pmatrix} \alpha I_{2\times2} & & 0_{2\times2} & \\ & B_{2\times2} & & \\ 0_{2\times2} & & \alpha I_{2\times2} & \\ & & & C_{2\times2} \end{pmatrix}, \tag{6.31}$$

where $0_{2\times 2}$ is the 2×2 matrix with all entries 0. For the $SU(4)_R$ indices $1,2,5,6$ this corresponds to a $U(1)$ symmetry assigning a charge of α to each index, while for the hidden indices (34) we assign $\beta \equiv Tr(B)$ and for (78) we assign $\gamma \equiv Tr(C)$. Once again we must take the tracelessness of the $SU(8)_R$ generators into account, implying the constraint $\gamma = -\beta - 4\alpha$. We can then assign the total charge to the component fields. For the Φ superfield we have

Helicity	KLT Product	Charge
$+2$	$(+1)\otimes(+1)$	0
$+\frac{3}{2}$	$(+\frac{1}{2}^{a_1})\otimes(+1)$, $(+1)\otimes(+\frac{1}{2}^{b_1})$	α
$+1$	$(0^{12})\otimes(+1)$, $(+\frac{1}{2}^{a_1})\otimes(+\frac{1}{2}^{b_1})$, $(+1)\otimes(0^{56})$	2α
$+\frac{1}{2}$	$(0^{12})\otimes(+\frac{1}{2}^{b_1})$, $(+\frac{1}{2}^{a_1})\otimes(0^{56})$	3α
0	$(0^{12})\otimes(0^{56})$	4α

where $a_1 = 1,2$ and $b_1 = 5,6$. For the Θ_{vector} superfield we have

Helicity	KLT Product	Charge
$+1$	$(0^{(34)})\otimes(+1)$	$0+\beta$
$+\frac{1}{2}$	$(0^{(34)})\otimes(+\frac{1}{2}^{b_1})$, $(-\frac{1}{2}^{a_1(34)})\otimes(+1)$	$\alpha+\beta$
0	$(0^{(34)})\otimes(0^{56})$, $(-\frac{1}{2}^{a_1(34)})\otimes(+\frac{1}{2}^{b_1})$, $(-1^{12(34)})\otimes(+1)$	$2\alpha+\beta$
$-\frac{1}{2}$	$(-\frac{1}{2}^{a_1(34)})\otimes(0^{56})$, $(-1^{12(34)})\otimes(+\frac{1}{2}^{b_1})$	$3\alpha+\beta$
-1	$(-1^{12(34)})\otimes(0^{56})$	$4\alpha+\beta$

For the Γ_{vector}-superfield

Helicity	KLT Product	Charge
$+1$	$(+1)\otimes(0^{(78)})$	$-4\alpha-\beta$
$+\frac{1}{2}$	$(+1)\otimes(-\frac{1}{2}^{b_1(78)})$, $(+\frac{1}{2}^{a_1})\otimes(0^{(78)})$	$-3\alpha-\beta$
0	$(0^{12})\otimes(0^{(78)})$, $(+\frac{1}{2}^{a_1})\otimes(-\frac{1}{2}^{b_1(78)})$, $(+1)\otimes(-1^{56(78)})$	$-2\alpha-\beta$
$-\frac{1}{2}$	$(0^{12})\otimes(-\frac{1}{2}^{b_1(78)})$, $(+\frac{1}{2}^{a_1})\otimes(-1^{56(78)})$	$-\alpha-\beta$
-1	$(0^{12})\otimes(-1^{56(78)})$	$0-\beta$

and finally for the Ψ-superfield

Helicity	KLT Product	Charge
0	$(0^{(34)})\otimes(0^{(78)})$	-4α
$-\frac{1}{2}$	$(0^{(34)})\otimes(-\frac{1}{2}^{b_1(78)})$, $(-\frac{1}{2}^{a_1(34)})\otimes(0^{(78)})$	-3α
-1	$(0^{(34)})\otimes(-1^{56(78)})$, $(-\frac{1}{2}^{a_1(34)})\otimes(-\frac{1}{2}^{b_1(78)})$, $(-1^{12(34)})\otimes(0^{(78)})$	-2α
$-\frac{3}{2}$	$(-\frac{1}{2}^{a_1(34)})\otimes(-1^{56(78)})$, $(-1^{12(34)})\otimes(-\frac{1}{2}^{b_1(78)})$	$-\alpha$
-2	$(-1^{12(34)})\otimes(-1^{56(78)})$	0

The matrix in eq. (6.31) represents two different classes of generators. The first one is given by setting $\beta = 0$. This assigns a different α charge to each component field within a supermultiplet, just like we also saw in the minimal $\mathcal{N}_G = 6$ supergravity case above. This corresponds to the additional $U(1)_R$ symmetry. The second possibility is to set $\alpha = 0$ in which case we can still assign a charge β (or $-\beta$) to the fields in the matter multiplets.

This indicates that there is yet another $U(1)$ symmetry, different from $U(1)_R$, acting on the matter multiplet only. The amplitudes in the $(\widetilde{\mathcal{N}}=2)\otimes(\mathcal{N}=2)$ gravity theory are therefore invariant under the $SU(4)_R\otimes U(1)_R\otimes U(1)$ group.

The extra $U(1)$ symmetries are needed to explain the vanishing of several of the amplitudes in the $(\widetilde{\mathcal{N}}=2)\otimes(\mathcal{N}=2)$ supergravity theory. For example, the amplitude

$$M_5^{\mathcal{N}_G=4}(v_+^{(34)}, v_+^{(34)}, h_-^{1256(3478)}, h_-^{1256(3478)}, h_+)\ , \tag{6.32}$$

does not violate the $SU(4)_R$ symmetry and conserves the $U(1)_R$ charge, but not the $U(1)$ charge (it has total charge 2β). Thus the violation of $U(1)$ ensures the vanishing of this amplitude. Similarly, the amplitude

$$M_5^{\mathcal{N}_G=4}(v_+^{(34)}, v_+^{(78)}, h_-^{1256(3478)}, h_-^{1256(3478)}, h_+)\ , \tag{6.33}$$

does not violate the $SU(4)_R$ symmetry, has zero $U(1)$ charge but the $U(1)_R$ charge is not conserved (its total charge is -4α), thus the violation of $U(1)_R$ implies the vanishing of this amplitude.

In a similar manner we find that the amplitudes of the $\mathcal{N}_G=0,1,2,3,4$ supergravity theories coupled to matter, which were constructed from products of minimal super-Yang-Mills theories, are enjoying $SU(\mathcal{N}_G)_R\otimes U(1)_R\otimes U(1)$ invariance. In the $\mathcal{N}_G=1$ case this reduces to invariance under $U(1)_R\otimes U(1)$ and in the $\mathcal{N}_G=0$ case further down to just $U(1)$ invariance.

Note that the extra (non-trivial) $U(1)$ symmetry was not possible in the case of minimal $\mathcal{N}_G=4,5,6$ supergravity, since setting $\alpha=0$ would also force β to be zero, due to the tracelessness of the matrix in eq. (6.27).

We summarise the above results in table 6.2, see also [28].

$\widetilde{\mathcal{N}} \otimes \mathcal{N}$	Number of states for component fields					Linear symmetry group from KLT product
	2	3/2	1	1/2	0	
$4 \otimes 4$	1	8	28	56	70	$SU(8)_R$
$4 \otimes 3$	1	7+1	21+7	35+21	35+35	$SU(8)_R$
$3 \otimes 3$	1	6+1+1	15+6+6+1	20+15+15+6	15+20+20+15	$SU(8)_R$
$4 \otimes 2$	1	6	15+1	20+6	15+15	$U(6)_R$
$3 \otimes 2$	1	5+1	10+5+1	10+10+5+1	5+10+10+5	$U(6)_R$
$4 \otimes 1$	1	5	10	10+1	5+5	$U(5)_R$
$3 \otimes 1$	1	4+1	6+4	4+6+1	1+4+4+1	$U(5)_R$
$4 \otimes 0$	1	4	6	4	1+1	$U(4)_R$
$3 \otimes 0$	1	3+1	3+3	1+3	1+1	$U(4)_R$
$2 \otimes 2$	1	4	6+1+1	4+4+4	1+6+6+1	$U(4)_R \otimes U(1)$
$2 \otimes 1$	1	3	3+1	1+3+1	3+3	$U(3)_R \otimes U(1)$
$2 \otimes 0$	1	2	1+1	2	1+1	$U(2)_R \otimes U(1)$
$1 \otimes 1$	1	2	1	1+1	2+2	$U(2)_R \otimes U(1)$
$1 \otimes 0$	1	1	0	1	1+1	$U(1)_R \otimes U(1)$
$0 \otimes 0$	1	0	0	0	1+1	$U(1)$

Table 6.2: Field content of the supergravity theories constructed from KLT-products, and their invariant linear groups as inferred from its embedding in maximally supersymmetric gravity. The total number of states for specific component fields is obtained by adding states in the different superfields (or diamonds) of the given theory. The linear global symmetry groups for minimal $4 \leq \mathcal{N}_G \leq 8$ supergravities are also listed in [75].

Chapter 7

Amplitude Relations at Loop Level

In this chapter we will review some of the progress that has been made in generalizing the above structures to loop level. It will by no means be a complete account of the developments that are happening in this direction of research. The goal of this chapter is to illustrate how an improved understanding of structures at tree level can guide us towards a better understanding of structures at loop level. We will focus on one-loop amplitude relations, and only make few comments at general loop order.

7.1 Color-Group Structures

Analogous to the full tree-level amplitudes in chapter 2, we can write down color decompositions for one-loop amplitudes. The standard one-loop n-point expression, with all particles in the adjoint representation, is given by

$$\begin{aligned}\mathcal{A}_n^{1-loop} = g^n \Bigg[& N \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}[T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}] A_{n;1}(\sigma(1),\ldots,\sigma(n)) \\ & + \sum_{c=2}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/S_{n;c}} \mathrm{Tr}[T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(c-1)}}]\mathrm{Tr}[T^{a_{\sigma(c)}}\cdots T^{a_{\sigma(n)}}] A_{n;c}(\sigma(1),\ldots,\sigma(n)) \Bigg],\end{aligned} \tag{7.1}$$

where $A_{n;1}$ are the planar or *leading-N* color-ordered amplitudes, and $A_{n;c>1}$ are the *subleading* amplitudes. The $\mathbb{Z}_n$ and $S_{n;c}$ are the subsets of S_n which respectively leave the single and double trace structure invariant. It is sufficient to know the $A_{n;1}$ amplitudes, since the subleading $A_{n;c>1}$ can all be obtained from

$$A_{n;c>1}(1,2,\ldots,c-1,c,c+1,\ldots,n) = (-1)^{c-1} \sum_{\sigma\in \mathrm{COP}(\{\alpha\}\cup\{\beta\})} A_{n;1}(\sigma(1),\ldots,\sigma(n))\,. \tag{7.2}$$

Here $\{\alpha\} \equiv \{c-1, c-2, \ldots, 2, 1\}$, $\{\beta\} \equiv \{c, c+1, \ldots, n-1, n\}$, and $\mathrm{COP}(\{\alpha\}\cup\{\beta\})$ is the set of all permutations of $\{\alpha\}\cup\{\beta\}$ that preserve the cyclic order of the elements within each set. These relations are the equivalence of the KK-relations at tree-level, and can be inferred from the color group by use of Jacobi identities in the same way.

Incorporating eq. (7.2) into eq. (7.1) an alternative color decomposition is obtained [13]

$$\mathcal{A}_n^{1-loop} = g^n \sum_{\sigma\in S_{n-1}/\mathcal{R}} \mathrm{Tr}[\widetilde{T}_{\mathrm{adj}}^{a_{\sigma(1)}}\cdots \widetilde{T}_{\mathrm{adj}}^{a_{\sigma(n)}}] A_{n;1}(\sigma(1),\ldots,\sigma(n))\,, \tag{7.3}$$

where $S_{n-1} \equiv S_n/\mathbb{Z}_n$ denotes non-cyclic permutations, which are moded out by reflections $\mathcal{R}$ in the sum, and we have defined the adjoint generators as $(\widetilde{T}^a_{\text{adj}})_{bc} \equiv i\widetilde{f}^{bac}$. Note that eq. (7.3) only contains the leading-N color-ordered amplitudes.

For later use we also present the corresponding decomposition in a theory with n_f flavors of quarks. Then the n-gluon one-loop amplitude is extended to [13]

$$\mathcal{A}'^{1-loop}_n = g^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} \Big[\text{Tr}[\widetilde{T}^{a_{\sigma(1)}}_{\text{adj}} \cdots \widetilde{T}^{a_{\sigma(n)}}_{\text{adj}}] A_{n;1}(\sigma(1), \ldots, \sigma(n)) \\ + 2n_f \text{Tr}[T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}] A^{[1/2]}_{n;1}(\sigma(1), \ldots, \sigma(n)) \Big] , \qquad (7.4)$$

where $A^{[1/2]}_{n;1}$ are the color-ordered gluon amplitudes with quarks circulating in the loop.

The color-ordered amplitudes $A_{n;1}$ and $A^{[1/2]}_{n;1}$ satisfy the same cyclic and reflection invariance as the tree-level amplitudes in (2.15).

This type of color decompositions can be extended to higher loops as well. The equivalence of eq. (7.1) will then include terms with a higher number of trace factors and corresponding subamplitudes. The constraints from the gauge group again imply relations among the subamplitudes, but in general they will not all be related to the leading subamplitudes [76–79].

In terms of the *integrated* subamplitudes no further general n-point relations are known. Especially a generalization of the BCJ-relations has not yet been achieved. At the *integrand* level some relations seem to reappear [80–82]. For instance, the loop momentum l can be chosen such that

$$0 = s_{2l} I(2,1,3,4,\ldots,n) + (s_{2l}+s_{12}) I(1,2,3,\ldots,n) + (s_{2l}+s_{12}+s_{23}) I(1,3,2,4,\ldots,n) \\ + \cdots + (s_{2l}+s_{12}+\cdots+s_{2(n-1)}) I(1,3,4,\ldots,n-1,2,n) , \qquad (7.5)$$

where I is the integrand of the planar one-loop subamplitude with the corresponding order of external legs. This expression is true up to terms that will vanish after loop integration. It is possible that this is the only place where such general structures survive.

However, there exists a special class of *finite* one-loop amplitudes. These are the ones with all or all but one helicity the same. Because of their "tree-like" appearance we restrict ourself to this subset of one-loop amplitudes and look for relations among them.

7.2 All-plus-helicity Amplitudes

Let us begin with the one-loop all-plus-helicity gluon amplitudes in Yang-Mills theory. The general n-point expression is given by [83]

$$A_{n;1}(1^+, 2^+, 3^+, \ldots, n^+) = -\frac{i}{48\pi^2} \frac{\sum_{i<j<k<l} \text{Tr}_-[ijkl]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \ldots \langle n1\rangle} , \qquad (7.6)$$

where $\text{Tr}_-[ijkl] \equiv \frac{1}{2}\text{Tr}[(1-\gamma_5)\not{p}_i\not{p}_j\not{p}_k\not{p}_l] = \langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle$.

For $n > 4$ these amplitudes satisfy the "triple-photon decoupling" identity

$$0 \;=\; \sum_{\sigma \in \mathcal{P}(\text{O}(\{\alpha\}) \cup \{\beta\})} A_{n;1}(1, \{\sigma\}) . \qquad (7.7)$$

In this expression $\{\alpha\}$ and $\{\beta\}$ are any partition of the legs $\{2, 3, \ldots, n\}$ ($n > 4$), with $\{\beta\}$ containing at least three elements, and the sum is over all permutations of leg $\{2, 3, \ldots, n\}$

with the order of the elements in the $\{\alpha\}$ set kept fixed. Note that here and for the rest of this section we will suppress the $+$ helicity label on the external legs.

Its name stems from the original case it was derived from. In ref. [83] it was found that for $n > 4$ and with three or more external photons the full all-plus-helicity one-loop amplitude in eq. (7.4) vanishes. If no fundamental particles were present the full amplitude would always vanish if one or more of the particles are photons. However, in this case we allowed quarks to circulate in the loop so the vanishing is not obvious. Since letting $T^a \to 1$ implies $(\widetilde{T}^a_{\text{adj}})_{bc} \to 0$, it follows directly from the result in [83] that

$$0 \;=\; \sum_{\sigma\in\mathcal{P}(\mathrm{O}(\{\alpha\})\cup\{\beta\})} A^{[1/2]}_{n;1}(1,\{\sigma\})\,. \tag{7.8}$$

Due to the supersymmetric Ward identity $A^{\text{SUSY}}(1^\pm,2^+,\ldots,n^+) = 0$, color-ordered amplitudes with quarks circulating in the loop are equal, up to a sign, to the color-ordered amplitudes with only gluons in the loop [37], and we thereby get eq. (7.7).

Let us give some explicit examples of the triple-photon decoupling identity. At five points the choices $\{\alpha\} = \{2\}$, $\{\beta\} = \{3,4,5\}$, or $\{\alpha\} = \emptyset$, $\{\beta\} = \{2,3,4,5\}$ result in trivial relations since all $(n-1)!$ permutations are summed over, and the amplitudes will just appear in canceling pairs $A_n(1,\sigma) + A_n(1,\sigma^T)$, by the reflection antisymmetry (2.15).

For $n \geq 6$ the relations are non-trivial. For example, at six points with $\{\alpha\} = \{2,3\}$ and $\{\beta\} = \{4,5,6\}$ we get

$$\begin{aligned}
0 = {}& A_{6;1}(1,2,3,4,5,6) + A_{6;1}(1,2,3,4,6,5) + A_{6;1}(1,2,3,5,4,6) + A_{6;1}(1,2,3,5,6,4)\\
&+ A_{6;1}(1,2,3,6,4,5) + A_{6;1}(1,2,3,6,5,4) + A_{6;1}(1,2,4,3,5,6) + A_{6;1}(1,2,4,3,6,5)\\
&+ A_{6;1}(1,2,4,5,3,6) + A_{6;1}(1,2,4,5,6,3) + A_{6;1}(1,2,4,6,3,5) + A_{6;1}(1,2,4,6,5,3)\\
&+ A_{6;1}(1,2,5,3,4,6) + A_{6;1}(1,2,5,3,6,4) + A_{6;1}(1,2,5,4,3,6) + A_{6;1}(1,2,5,4,6,3)\\
&+ A_{6;1}(1,2,5,6,3,4) + A_{6;1}(1,2,5,6,4,3) + A_{6;1}(1,2,6,3,4,5) + A_{6;1}(1,2,6,3,5,4)\\
&+ A_{6;1}(1,2,6,4,3,5) + A_{6;1}(1,2,6,4,5,3) + A_{6;1}(1,2,6,5,3,4) + A_{6;1}(1,2,6,5,4,3)\\
&+ A_{6;1}(1,4,2,3,5,6) + A_{6;1}(1,4,2,3,6,5) + A_{6;1}(1,4,2,5,3,6) + A_{6;1}(1,4,2,5,6,3)\\
&+ A_{6;1}(1,4,2,6,3,5) + A_{6;1}(1,4,2,6,5,3) + A_{6;1}(1,4,5,2,3,6) + A_{6;1}(1,4,5,2,6,3)\\
&+ A_{6;1}(1,4,5,6,2,3) + A_{6;1}(1,4,6,2,3,5) + A_{6;1}(1,4,6,2,5,3) + A_{6;1}(1,4,6,5,2,3)\\
&+ A_{6;1}(1,5,2,3,4,6) + A_{6;1}(1,5,2,3,6,4) + A_{6;1}(1,5,2,4,3,6) + A_{6;1}(1,5,2,4,6,3)\\
&+ A_{6;1}(1,5,2,6,3,4) + A_{6;1}(1,5,2,6,4,3) + A_{6;1}(1,5,4,2,3,6) + A_{6;1}(1,5,4,2,6,3)\\
&+ A_{6;1}(1,5,4,6,2,3) + A_{6;1}(1,5,6,2,3,4) + A_{6;1}(1,5,6,2,4,3) + A_{6;1}(1,5,6,4,2,3)\\
&+ A_{6;1}(1,6,2,3,4,5) + A_{6;1}(1,6,2,3,5,4) + A_{6;1}(1,6,2,4,3,5) + A_{6;1}(1,6,2,4,5,3)\\
&+ A_{6;1}(1,6,2,5,3,4) + A_{6;1}(1,6,2,5,4,3) + A_{6;1}(1,6,4,2,3,5) + A_{6;1}(1,6,4,2,5,3)\\
&+ A_{6;1}(1,6,4,5,2,3) + A_{6;1}(1,6,5,2,3,4) + A_{6;1}(1,6,5,2,4,3) + A_{6;1}(1,6,5,4,2,3)\,.
\end{aligned} \tag{7.9}$$

This relation contains all 60 amplitudes that are not related by cyclicity or reflection. The choices, $\{\alpha\} = \{2\}, \{\beta\} = \{3,4,5,6\}$, and $\{\alpha\} = \emptyset$, $\{\beta\} = \{2,3,4,5,6\}$ also result in valid relations. However, these are not independent of the relations in which $\{\beta\}$ contains exactly 3 elements. They will just be sums of relations of the form given in eq. (7.9).

At seven points, the case of $\{\alpha\} = \{2,3,4\}$ and $\{\beta\} = \{5,6,7\}$ leads to a relation containing 120 amplitudes. Again, the remaining relations obtained from $\{\alpha\} = \{2,3\}$, $\{\beta\} = \{4,5,6,7\}$ and $\{\alpha\} = \{2\}$, $\{\beta\} = \{3,4,5,6,7\}$ and $\{\alpha\} = \emptyset$, $\{\beta\} = \{2,3,4,5,6,7\}$ are not independent of the identities where $\{\beta\}$ contains exactly 3 elements, but just sums of relations with this form.

As part of a search for additional relations among the all-plus-helicity amplitudes [84], a diagrammatic representation, very similar to the one we discussed in section 3.2, was introduced.

Diagrammatic Representation

It has been conjectured that any one-loop all-plus-helicity amplitude can be represented as a sum over all distinct planar diagrams, where each diagram contains exactly *one* totally symmetric quartic vertex D^{abcd} and all remaining legs are attached by means of anti-symmetric cubic vertices F^{abc} in all possible planar distinct ways [84].

From this conjecture we, for instance, represents the four-point amplitude by a single symmetric quartic vertex

$$A_{4;1}(1,2,3,4) = D^{a_1a_2a_3a_4}\,. \tag{7.10}$$

The explicit expression, see eq. (7.6), for this amplitude is

$$A_{4;1}(1,2,3,4) = -\frac{i}{48\pi^2}\frac{[14][23]}{\langle 14\rangle\langle 23\rangle}\,. \tag{7.11}$$

Using momentum conservation it is straightforward to check that it is indeed totally crossing symmetric, *e.g.*

$$\frac{[14][23]}{\langle 14\rangle\langle 23\rangle} = \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} = \frac{[13][24]}{\langle 13\rangle\langle 24\rangle}\,, \tag{7.12}$$

consistent with the conjecture.

For the five-point amplitude $A_{5;1}$ the conjecture implies the following representation

$$\begin{aligned} A_{5;1}(1,2,3,4,5) = {} & D^{a_1a_2a_3b}F^{ba_4a_5} + D^{a_2a_3a_4b}F^{ba_5a_1} + D^{a_3a_4a_5b}F^{ba_1a_2} \\ & + D^{a_4a_5a_1b}F^{ba_2a_3} + D^{a_5a_1a_2b}F^{ba_3a_4}\,, \end{aligned} \tag{7.13}$$

shown diagrammatically in figure 7.1. Note that the purpose of the indices on the vertices is only to indicate the connection between the vertices, and the position of the external legs in the diagrams.

As a non-trivial consistency check the reflection symmetry can be derived from this diagrammatic expansion

$$\begin{aligned} A_{5;1}(1,2,3,4,5) = {} & -D^{a_1a_2a_3b}F^{ba_5a_4} - D^{a_2a_3a_4b}F^{ba_1a_5} - D^{a_3a_4a_5b}F^{ba_2a_1} \\ & - D^{a_4a_5a_1b}F^{ba_3a_2} - D^{a_5a_1a_2b}F^{ba_4a_3} \\ = {} & -\big(D^{a_3a_2a_1b}F^{ba_5a_4} + D^{a_4a_3a_2b}F^{ba_1a_5} + D^{a_5a_4a_3b}F^{ba_2a_1} \\ & + D^{a_1a_5a_4b}F^{ba_3a_2} + D^{a_2a_1a_5b}F^{ba_4a_3}\big) \\ = {} & -A_{5;1}(5,4,3,2,1)\,. \end{aligned} \tag{7.14}$$

The cyclicity is obvious because of the sum over all different planar diagrams. For higher n-point amplitudes this antisymmetry under reflection (or symmetry, for even n) immediately generalizes using the diagrammatic representation.

In this way it is possible to build up a diagrammatic representation for any n-point all-plus-helicity amplitude. Similarly to how the KK-relations can be inferred from the BCJ-representation at tree-level using only the antisymmetry of the cubic vertices, this representation can be used to look for relations among the all-plus-helicity amplitudes. Let us present some of the relations obtained from such investigations. Note that all of the following relations were checked with the explicit expression from eq. (7.6) up to at least 15 points [84].

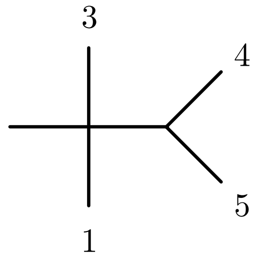
$A_{5;1}(1,2,3,4,5) \quad = \quad 2$  $+$ *cyclic perm.*

Figure 7.1: Diagrammatic representation of the five-point one-loop all-plus-helicity amplitude.

Additional All-plus-helicity Amplitude Relations

At four points we have already argued for the crossing symmetry, *i.e.*

$$A_{4;1}(1,2,3,4) = A_{4;1}(1,3,4,2) = A_{4;1}(1,4,2,3)\,. \tag{7.15}$$

At five points we have

$$\begin{aligned}0 = {} & A_{5;1}(1,4,3,5,2) + A_{5;1}(1,5,3,4,2) + A_{5;1}(1,2,3,4,5) \\ & + A_{5;1}(1,2,3,5,4) + A_{5;1}(1,5,3,2,4) + A_{5;1}(1,4,3,2,5)\,.\end{aligned} \tag{7.16}$$

It is easy to check that this follows from the diagrammatic representation in eq. (7.13). For instance, there is a diagram $D^{a_1a_2a_3b}F^{ba_4a_5}$ from both $A_{5;1}(1,2,3,4,5)$ and $A_{5;1}(1,2,3,5,4)$, but since $F^{ba_4a_5}$ is antisymmetric in $4 \leftrightarrow 5$ the diagrams cancels out in the sum. Similarly each diagram gets canceled out in (7.16).

Eq. (7.16) can be rewritten in the form

$$\begin{aligned}0 = {} & 2A_{5;1}(1,2,3,4,5) + (-1)^2 \sum_{\sigma\in \mathrm{OP}(\{4\}\cup\{2,1\})} A_{5;1}(3,\{\sigma\},5) \\ & + \mathcal{P}(4,5)\,,\end{aligned} \tag{7.17}$$

which generalizes to an arbitrary n-point expression as

$$\begin{aligned}0 = {} & 2A_{n;1}(1,2,\ldots,n) + (-1)^{n-3} \sum_{\sigma\in \mathrm{OP}(\{4\}\cup\{\beta\})} A_{n;1}(3,\{\sigma\},5) \\ & + \mathcal{P}(4,5,\ldots,n)\,,\end{aligned} \tag{7.18}$$

with $\{\beta\} = \{2,1,n,n-1,\ldots,6\}$.

Another (independent) n-point relation is

$$0 = 6A_{n;1}(1,2,\ldots,n) - \sum_{k=2}^{n-1}\left[\sum_{\sigma_k\in \mathrm{OP}(\{\alpha_k\}\cup\{\beta_k\})} A_{n;1}(1,\{\sigma_k\})\right], \tag{7.19}$$

where $\{\alpha_k\} \equiv \{2,3,\ldots,k\}$ and $\{\beta_k\} \equiv \{k+1,\ldots,n\}$ and OP is the set of "ordered permutations" obtained from the two sets.

For example, at five points this is

$$\begin{aligned}0 = {} & 3A_{5;1}(1,2,3,4,5) - A_{5;1}(1,2,3,5,4) - A_{5;1}(1,2,4,3,5) - A_{5;1}(1,2,4,5,3) \\ & - A_{5;1}(1,2,5,3,4) - A_{5;1}(1,3,2,4,5) - A_{5;1}(1,3,4,2,5) - A_{5;1}(1,3,4,5,2) \\ & - A_{5;1}(1,4,2,3,5) - A_{5;1}(1,4,2,5,3) - A_{5;1}(1,4,5,2,3) - A_{5;1}(1,5,2,3,4)\,.\end{aligned} \tag{7.20}$$

For n-point functions there are $2^{n-1}-n+1$ terms in eq. (7.19); the only exception being the case of $n = 8$, which has only $2^{n-1} - n$ terms. The coefficient of the first amplitude $A_{n;1}(1,2,\ldots,n)$ is $6-(n-2) = (8-n)$, and thus at precisely $n = 8$ this term is not included in the sum. All other coefficients in the sum are one.

7.3 One-minus-helicity Amplitudes

The other type of finite one-loop amplitudes are those with *one* minus helicity and the rest positive. Their explicit form is more complicated than for the all-plus amplitudes. For instance, the four- and five-point amplitudes are [85–87]

$$A_{4;1}(1^-,2^+,3^+,4^+) = \frac{i}{3}\frac{\langle 24\rangle[24]^3}{[12]\langle 23\rangle\langle 34\rangle[41]}\,, \tag{7.21}$$

and

$$A_{5;1}(1^-,2^+,3^+,4^+,5^+) = \frac{i}{3}\frac{1}{\langle 34\rangle^2}\left[-\frac{[25]^3}{[12][51]} + \frac{\langle 14\rangle^3[45]\langle 35\rangle}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2} - \frac{\langle 13\rangle^3[32]\langle 42\rangle}{\langle 15\rangle\langle 54\rangle\langle 32\rangle^2}\right]. \tag{7.22}$$

The higher-point amplitudes get increasingly complicated [88, 89], but still turn out to satisfy some tree-like relations.

The main guiding principle in finding relations for this class of amplitudes come from the following observation [84]:
Amplitude relations which are valid for *both* tree-level amplitudes and all-plus-helicity one-loop amplitudes, seem to automatically also be satisfied by the one-minus one-loop amplitudes.

From this observation it was realised that the triple-photon decoupling relation (7.7) is also valid for one-minus amplitudes. As discussed above, these relations hold for the all-plus amplitudes and also for tree amplitudes, since these satisfy single-photon decoupling relations (which imply that any number of photons decouple).

It was further found that BCJ-like relations can be written down as well.

Additional One-minus-helicity Amplitude Relations

Let us first define

$$\mathcal{Q}_n(1,2,\ldots,n) \equiv 6A_n(1,2,\ldots,n) - \sum_{k=2}^{n-1}\left[\sum_{\sigma_k\in \mathrm{OP}(\{\alpha_k\}\cup\{\beta_k\})} A_n(1,\{\sigma_k\})\right], \tag{7.23}$$

where the ordering of $\mathcal{Q}_n$ is simply dictated by the ordering of the first amplitude on the right-hand side. We have not yet specified which amplitudes the A_n's stand for, (*i.e.* whether tree-level, one-loop all-plus or one-loop one-minus). When the A_n's are one-loop all-plus amplitudes this is nothing but eq. (7.19), so it follows $\mathcal{Q}_n^{\mathrm{all-plus}}(1,2,\ldots,n) = 0$. When we take the amplitudes in eq. (7.23) to be of tree level, the expression inside the bracket vanishes, and we are left with

$$\mathcal{Q}_n^{\mathrm{tree}}(1,2,\ldots,n) = 6A_n^{\mathrm{tree}}(1,2,\ldots,n)\,. \tag{7.24}$$

From the BCJ-relations in eq. (3.27) we then know that

$$\begin{aligned}0 = {}& s_{12}\mathcal{Q}_n(1,2,3,\ldots,n) + (s_{12}+s_{23})\mathcal{Q}_n(1,3,2,4,\ldots,n)\\ &+ (s_{12}+s_{23}+s_{24})\mathcal{Q}_n(1,3,4,2,5,\ldots,n) + \cdots\\ &+ (s_{12}+s_{23}+s_{24}+\cdots+s_{2(n-1)})\mathcal{Q}_n(1,3,4,\ldots,n-1,2,n)\,,\end{aligned} \tag{7.25}$$

is a relation which is satisfied simultaneously by all-plus one-loop amplitudes (trivially, since each $\mathcal{Q}_n$ is zero here) and tree-level amplitudes (on account of tree-level BCJ-relations). In consistency with the observation, this relation turned out to also be satisfied by the one-minus-helicity amplitudes, checked numerically up to ten points in [84].

7.4 BCJ-Representation at Loop Level

Remarkably, there are strong indications that the color-kinematic duality, and even the connection to gravity through "squaring" of numerators, can be extended to loop level [90–97]. The idea is that the n-point gauge-theory amplitude at L loops can be written in the form

$$(-i)^L A_n^{L-loop} = \sum_i \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{c_i n_i}{(\prod_j s_j)_i} , \tag{7.26}$$

where the sum is over all n-point L-loop diagrams with antisymmetric cubic vertices. S_i is a symmetry factor of diagram i, and the numerators n_i again satisfy the color-kinematic duality in (3.80). Notice that the numerators belong to the integrand, and since the BCJ-relations at tree-level could be seen as a consequence of such a Jacobi structure, this might explain why it has proven so hard to extend to general relations at the integrated level.

The squaring of gauge-theory numerators to obtain gravity amplitudes has also been conjectured to apply at loop level, *i.e.* the n-point L-loop gravity amplitude should be

$$(-i)^{L+1} M_n^{L-loop} = \sum_i \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{\widetilde{n}_i n_i}{(\prod_j s_j)_i} , \tag{7.27}$$

with the same sum as in the gauge theory case. This is not only a theoretically interesting extension of the connection between perturbative gravity and gauge theory beyond tree level, but together with generalized unitarity cut methods [98] can have significant implications for multi-loop calculations, see *e.g.* [92]. It has been used successfully to show the finiteness of $\mathcal{N}_G = 8$ supergravity up to (as for now) four loops [99].

Chapter 8

Summary and Final Comments

In this thesis we examined the structure of scattering amplitude relations in gauge and gravity theories with and without supersymmetry.

After a short historic introduction into the field, we reviewed some aspects of scattering amplitudes which would be important in later chapters. This introduced us to necessary theoretical tools and illustrated how unexpected the existence of *any* connection between perturbative gravity and gauge theories is from the traditional Feynman diagrammatic point of view. This is especially surprising since gauge theory includes only cubic and quartic vertices, while perturbative gravity has an infinite amount of vertices. In addition, even cubic and quartic gravity vertices seem far more complicated than gauge-theory vertices. However, explicit calculations unveil amazing simplifications. Going on-shell (with complex momenta) the three-point gravity amplitude can be written as the square of the three-point gauge-theory amplitude

$$M_3(1,2,3) = A_3(1,2,3)^2\,. \tag{8.1}$$

The importance of this relation was further emphasized by the introduction of the BCFW recursion relation, which applies for both gauge and gravity amplitudes. It allows us to construct an arbitrary n-point amplitude using the three-point amplitudes as seeds.

In chapter 2 we also reviewed relations between color-ordered gauge-theory subamplitudes that have been known for a long time. They all follow from the color decomposition and properties of the gauge group, and reduce the total number of independent subamplitudes to $(n-2)!$.

In chapter 3 we looked at a more recent class of subamplitude relations. Although they were first discovered in field theory, where they are known as BCJ-relations, the first proof for them was obtained using string theory. By flipping integration contours through the complex plane, general open-string amplitude relations like

$$\begin{aligned}0 = \mathcal{A}_n(2,1,3,\ldots,n) &+ e^{i\pi\alpha' k_2\cdot k_1}\mathcal{A}_n(1,2,3,\ldots,n) + e^{i\pi\alpha' k_2\cdot(k_1+k_3)}\mathcal{A}_n(1,3,2,4,\ldots,n)\\ &+ \cdots + e^{i\pi\alpha' k_2\cdot(k_1+k_3+\cdots+k_{n-1})}\mathcal{A}_n(1,3,4,\ldots,n-1,2,n)\,,\end{aligned} \tag{8.2}$$

can be obtained. The real part gives a string-theory generalization of the Kleiss-Kuijf relations known from field theory, while the imaginary part provides the string version of the BCJ-relations. In the field-theory limit these expressions reduce exactly to the KK- and BCJ-relations, respectively. Both in string and field theory they reduce the number of independent subamplitudes down to $(n-3)!$.

We also showed how the BCJ-relations can be derived from field theory using the BCFW recursion relation. This is done starting from a contour integral which is equal to zero, but, through the gauge-group relations and lower-point BCJ-relations, can be rewritten into an expression similar to the field theory limit of the imaginary part of eq. (8.2). In this way the BCJ-relations can be obtained recursively.

We also saw how these relations can be seen as a consequence of a color-kinematic duality, writing gauge-theory amplitudes in terms of diagrams with only antisymmetric cubic vertices. This can be done in such a way, that the kinematic numerators obtained from the diagrams satisfy the same Jacobi identities as the corresponding color-factors. This is the way the BCJ-relations were originally discovered.

After examining the monodromy and BCJ-relations, in chapter 4 we turned to the factorization of closed-string amplitudes. Here we rederived the KLT-relations, using the same method as in the original paper, however, writing them in a slightly more explicit n-point form. This form nicely captures many of the equivalent, but different looking, expressions that exist for KLT-relations. These follow from the different choices one has in closing integration contours.

For a similar kind of factorization of states in amplitudes with closed-strings/gravitons coupled to open-strings/gluons see [16, 100–102], and for an analysis of the KLT-relations at high energies see [103].

In the field-theory limit the KLT-relations connect graviton amplitudes to products of gluon amplitudes as

$$\begin{aligned} M_n &= (-1)^{n+1}\times \\ &\sum_{\sigma}\sum_{\gamma,\beta}\mathcal{S}[\gamma(\sigma(2),\ldots,\sigma(j-1))|\sigma(2,\ldots,j-1)]_{k_1}\mathcal{S}[\sigma(j),\ldots,\sigma(n-2)|\beta(\sigma(j,\ldots,n-2))]_{k_{n-1}} \\ &\times A_n(1,\sigma(2,\ldots,n-2),n-1,n)\,\widetilde{A}_n(\gamma(\sigma(2),\ldots,\sigma(j-1)),1,n-1,\beta(\sigma(j),\ldots,\sigma(n-2)),n)\,, \end{aligned} \tag{8.3}$$

with $2\leq j\leq n-1$, that can be chosen freely. For instance, for $j=n-1$ we get the compact and beautiful n-point expression

$$M_n = (-1)^{n+1}\sum_{\sigma,\widetilde{\sigma}\in S_{n-3}}\widetilde{A}_n(n-1,n,\widetilde{\sigma}_{2,n-2},1)\mathcal{S}[\widetilde{\sigma}_{2,n-2}|\sigma_{2,n-2}]_{k_1}A_n(1,\sigma_{2,n-2},n-1,n)\,. \tag{8.4}$$

These relations are highly non-trivial in several aspects; first of all, from the Feynman diagram perspective gravity looks much worse than gauge theory (even than gauge theory “squared”, see *e.g.* [104]). Secondly, the gauge-theory side does not obviously exhibit permutational invariance in all external legs. And thirdly, the product of gauge-theory amplitudes will lead to double-poles which have to be canceled out precisely by the momentum kernel, such that only single-poles are present. Not even the intuitive picture of “gluing” two open strings together to form a closed string makes much sense from the field theoretical point of view. Inspired by the squaring relation at three points and the recursive structure exhibited by amplitudes through the BCFW recursion relations, we set out to prove the KLT-relations in pure field theory. Beside the BCFW and the three-point squaring relation, the proof relied on three main ingredients.

The first were the BCJ-relations. From these we could show that the right-hand side of eq. (8.3) was independent of j. This could have been expected since the freedom of j in the string derivation came from how we chose to close the integration contours. This was very reminiscent of the way we derived the monodromy relations by flipping contours, which was exactly the string-theory counterpart of BCJ-relations. The close connection between the BCJ-relations and the different expressions for the KLT-relations was further stressed by relations like

$$0 = \sum_{\sigma \in S_{n-2}} \mathcal{S}[\beta(2,\ldots,n-1)|\sigma(2,\ldots,n-1)]_{k_1} A_n(1,\sigma(2,\ldots,n-1),n)\,. \tag{8.5}$$

These were also useful directly in the proof of the KLT-relations.

The second important feature were the "vanishing identities". These state that when the KLT-product is taken between gauge-theory amplitudes belonging to different helicity sectors, the resulting expression is zero, *i.e.*

$$0 = \sum_{\sigma,\widetilde{\sigma} \in S_{n-3}} \widetilde{A}_n^k(n-1,n,\widetilde{\sigma}_{2,n-2},1)\mathcal{S}[\widetilde{\sigma}_{2,n-2}|\sigma_{2,n-2}]_{k_1} A_n^{k'}(1,\sigma_{2,n-2},n-1,n)\,, \tag{8.6}$$

when $k \neq k'$, where k and k' denoted the helicity sectors by N^kMHV and $N^{k'}MHV$. This can be proven recursively, but is more naturally explained from a maximally supersymmetric point of view. The KLT-products which vanish are exactly those that would correspond to gravity amplitudes violating the $SU(8)_R$ R-symmetry. This discussion and related arguments for less than maximal supersymmetry were picked up again in chapter 6.

The final piece was the introduction of an alternative version of the KLT-relations. This version is only well-defined when one leg is taken off-shell, and the gravity amplitude can then be obtained from the on-shell limit

$$M_n = (-1)^n \lim_{x\to 0} \sum_{\sigma,\widetilde{\sigma} \in S_{n-2}} \frac{\widetilde{A}_n(n',\widetilde{\sigma}_{2,n-1},1')\mathcal{S}[\widetilde{\sigma}_{2,n-1}|\sigma_{2,n-1}]_{p_1'} A_n(1',\sigma_{2,n-1},n')}{s_{1'2\ldots(n-1)}}\,. \tag{8.7}$$

Although this expression looks slightly odd, it turned out to be more straightforward than eq. (8.3) to prove. We also illustrated how it follows from the soft-limit behaviour of gravity amplitudes if eq. (8.4) is assumed.

After discussing/proving the above properties we gave a prove first of eq. (8.7) and then of eq. (8.3). Everything used in this derivation followed from very general properties of tree amplitudes. This showed that the KLT-relations can be seen as a consequence of the constraints such analytical properties put on amplitudes.

The KLT-relations are not restricted to pure Einstein gravity and Yang-Mills theory. In chapter 6 we showed how they extend to relations between $\mathcal{N}_G = 8$ supergravity and $\mathcal{N} = 4$ super-Yang-Mills theory. In terms of superamplitudes they can be written as one compact expression which encodes all the component KLT-relations. This form also includes the vanishing identities. We further showed how one can reduce this to less than maximal supersymmetric theories. We especially investigated which gravity theories the KLT-product maps into when the degree of supersymmetry on the gauge-theory side is lowered. From the less than maximally supersymmetric point of view we saw how the vanishing KLT-products are no longer just a consequence of $SU(\mathcal{N}_G < 8)_R$ violation, but also follow from breaking of $U(1)$-charge conservation. In minimal supergravity theories the $U(1)_R$ charge arises from the R-symmetry group $U(\mathcal{N}_G)_R = U(1)_R \otimes SU(\mathcal{N}_G)_R$, while in the cases with gravity coupled to matter there is an additional $U(1)$ symmetry acting on the matter-multiplet.

Finally, in chapter 7 we looked at how some of these tree-level results carry over to loop level. This included the Kleiss-Kuijf relations, which at one-loop level express subleading amplitudes $A_{n;c>1}$ in terms of leading-N amplitudes $A_{n;1}$. We then restricted our analysis to the finite one-loop amplitudes which satisfy further KK- and BCJ-like relations, and have diagrammatic representations very similar to those introduced by BCJ at tree-level.

We also mentioned a very interesting generalization of the BCJ-parametrization to loop level. The claim is that as for tree-level amplitudes the integrand of an arbitrary L-loop amplitude can be written in terms of numerators satisfying the color-kinematic duality. This leads to a possible extension of the KLT-relations to loops, where the squaring occurs at the numerators in the integrand.

Since the kinematic numerators satisfy Jacobi identities, it has been speculated if there exists an algebra for the kinematic part of scattering amplitudes, see *e.g.* [61]. This has been found for the self-dual sector of Yang-Mills theory, where it appears as an area-preserving diffeomorphism Lie algebra [62]. It has also been shown how one can systematically express amplitudes in terms of auxiliary amplitudes with an underlying kinematic algebra [63]. However, it is still an unsolved problem if an algebra exists from which BCJ-numerators for a general amplitude can be directly obtained. This would also be very interesting for gravity amplitudes due to their connection to the square of such numerators.

Another interesting line of developments, which has recently had a breakthrough, is how one can write compact expression for gravity amplitudes. In ref. [105] Hodges gave a compact formula expressing a n-point MHV graviton amplitude in terms of a determinant of a $(n-3) \times (n-3)$ matrix. This formula can be directly related to the one we presented in eq. (2.58) using graph theory [106]. Furthermore it inspired the construction of a $\mathcal{N}_G = 8$ supergravity version [107] of the Roiban-Spradlin-Volovich-Witten (RSVW) formula for $\mathcal{N} = 4$ super-Yang-Mills tree amplitudes [4,108]. In constructing this new gravity formula it was observed that the individual terms from the RSVW formula, making up the super-Yang-Mills amplitude, behave very much like color-ordered subamplitudes. In particular, beside satisfying the KK- and BCJ-relations [109], they also seem to satisfy a type of vanishing identity. In the RSVW-formula the amplitude is made up of a MHV-like expression evaluated on the support of certain delta-functions. It is thereby expressed as a sum over the solutions to the argument of these delta-functions (times a factor from the corresponding Jacobian). The observation in [107] was that the KLT-product between the MHV-like expressions evaluated at *different* solutions vanishes, even if the solutions belong to the *same* helicity sector. Understanding this structure better, might reveal further insight about the connection between gravity and gauge theories we have considered in this thesis.

It is very hard to predict even the nearest future within the field of scattering amplitudes. Developments are happening faster than even the most optimistic people would have thought just a few years back. New ideas and methods spring to life almost monthly, so today's impossible problems can very well be tomorrows trivial exercises.

Acknowledgements

I would like to thank my advisors Poul Henrik Damgaard and Emil Bjerrum-Bohr for numerous discussions, good collaboration on exciting projects and their invaluable help and advice during my studies.

I would also like to thank Bo Feng and Pierre Vanhove for our fruitful collaborations and many stimulating discussions.

I am very grateful to the Kavli Institute of Theoretical Physics at University of California, Santa Barbara, and to Zvi Bern for his helpful advice during my stay at KITP.

I would like to thank all my colleges at the Niels Bohr Institute and Niels Bohr International Academy for an always great academic and social atmosphere. Special thanks go to Lisa Glaser for comments and useful suggestions to an early draft of this thesis.

I am also very thankful to the staff at the Niels Bohr Institute and especially Helle Kiilerich for her much appreciated help on several occasions.

Finally I would like to thank all the amazing people I have had the pleasure to meet during my studies, making the last three years some of the most exciting in my life.

Appendix A

The Complex Power Function and Branch Cuts

The complex power function z^c is in general a *multi-valued* function. As a simple example consider $z^{1/2}$ and take $z = |z|e^{i\theta} = |z|e^{i\theta+2\pi i}$, then

$$\begin{aligned} z^{1/2} &= \left(|z|e^{i\theta}\right)^{1/2} = |z|^{1/2}e^{i\theta/2}\,, \\ z^{1/2} &= \left(|z|e^{i\theta+2\pi i}\right)^{1/2} = |z|^{1/2}e^{i\theta/2}e^{i\pi} = -|z|^{1/2}e^{i\theta/2}\,, \end{aligned} \tag{A.1}$$

which are two different values for the same z.

In order to have a well-defined function, here meaning *single-valued*, we need to impose a *branch cut* on the complex plane. A common choice is to place the branch cut on the negative real axis, *i.e.* restricting the power function to

$$z^c = |z|^c e^{ci\theta}\,, \qquad -\pi < \theta \leq \pi\,. \tag{A.2}$$

This means that the complex number $z = |z|e^{i\theta}$ should always be represented with θ lying in $]-\pi,\pi]$ before taking it to the power c.

A.1 Phase Factors

Let us see how the above branch cut dictates the phase factors in eq. (4.17). That is, considering eq. (A.2) for some z_0 with $\mathrm{Re}(z_0) < 0$. If $\mathrm{Im}(z_0) \geq 0$ we have

$$(z_0)^c = |z_0|^c e^{ci\theta_0}\,, \qquad \pi/2 < \theta_0 \leq \pi\,. \tag{A.3}$$

If we then look at the power of $-z_0$ we get

$$(-z_0)^c = |z_0|^c e^{ci(\theta_0-\pi)} = e^{-i\pi c}|z_0|^c e^{ci\theta_0} = e^{-i\pi c}(z_0)^c\,. \tag{A.4}$$

We had to take the argument for $(-z_0)$ as $\theta_0 - \pi$ in order to not fall outside the $]-\pi,\pi]$ region. We conclude that

$$(z_0)^c = e^{i\pi c}(-z_0)^c\,. \tag{A.5}$$

Likewise, if $\mathrm{Im}(z_0) < 0$ we have

$$(z_0)^c = |z_0|^c e^{ci\theta_0}\,, \qquad -\pi < \theta_0 < -\pi/2\,, \tag{A.6}$$

and for $-z_0$

$$(-z_0)^c = |z_0|^c e^{ci(\theta_0+\pi)} = e^{i\pi c}|z_0|^c e^{ci\theta_0} = e^{i\pi c}(z_0)^c\,, \tag{A.7}$$

hence

$$(z_0)^c = e^{-i\pi c}(-z_0)^c\,. \tag{A.8}$$

If we instead place the branch cut on the positive real axis we find the phase factors corresponding to eq. (4.18).

The careful reader might also have noticed that in eq. (4.10) and (4.11) we used $(z_1 z_2)^c = z_1^c z_2^c$. This is allowed here without any modifications in terms of phase factors because z_1 and z_2 had opposite sign on their imaginary part. For instance, say $\mathrm{Im}(z_1) < 0$ and $\mathrm{Im}(z_2) > 0$ with the branch cut on the negative real axis. Then $\theta_1 \in]-\pi, 0[$ and $\theta_2 \in]0, \pi[$, which imply $(\theta_1 + \theta_2) \in]-\pi, \pi[$, and therefore in this situation $z_1^c z_2^c$ is equal to $(z_1 z_2)^c$.

Appendix B

The Shifting-Formula for j

Here we provide a detailed proof of eq. (5.15). We begin with the following rewriting

$$
\begin{aligned}
&\sum_{\alpha} \mathcal{S}[\alpha_{i_2,i_j}|i_2,\ldots,i_j]_{p_1} \widetilde{A}_n(\alpha_{i_2,i_j},1,n-1,i_{j+1},\ldots,i_{n-2},n) \\
&= \sum_{\alpha'} \Big[\mathcal{S}[\alpha'_{i_2,i_{j-1}},i_j|i_2,\ldots,i_j]_{p_1} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},i_j,1,n-1,i_{j+1},\ldots,i_{n-2},n) \\
&+ \mathcal{S}[\alpha'_{i_2,i_{j-2}},i_j,\alpha'(i_{j-1})|i_2,\ldots,i_j]_{p_1} \widetilde{A}_n(\alpha'_{i_2,i_{j-2}},i_j,\alpha'(i_{j-1}),1,n-1,i_{j+1},\ldots,i_{n-2},n) \\
&+ \cdots + \mathcal{S}[i_j,\alpha'_{i_2,i_{j-1}}|i_2,\ldots,i_j]_{p_1} \widetilde{A}_n(i_j,\alpha'_{i_2,i_{j-1}},1,n-1,i_{j+1},\ldots,i_{n-2},n) \Big] \\
&= \sum_{\alpha'} \mathcal{S}[\alpha'_{i_2,i_{j-1}}|i_2,\ldots,i_{j-1}]_{p_1} \times \Big[s_{1j} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},i_j,1,n-1,i_{j+1},\ldots,i_{n-2},n) \\
&+ (s_{1j}+s_{j\alpha'(j-1)}) \widetilde{A}_n(\alpha'_{i_2,i_{j-2}},i_j,\alpha'(i_{j-1}),1,n-1,i_{j+1},\ldots,i_{n-2},n) + \cdots \\
&+ (s_{1j}+s_{j\alpha'(1)}+s_{j\alpha'(2)}+\cdots+s_{j\alpha'(j-1)}) \widetilde{A}_n(i_j,\alpha'_{i_2,i_{j-1}},1,n-1,i_{j+1},\ldots,i_{n-2},n) \Big] .
\end{aligned}
\tag{B.1}
$$

Using the fundamental BCJ-relation on the expression inside $[\cdots]$ as well as momentum conservation, we get

$$
\begin{aligned}
&\sum_{\alpha'} \mathcal{S}[\alpha'_{i_2,i_{j-1}}|i_2,\ldots,i_{j-1}]_{p_1} \\
&\times \Big[(s_{j(n-1)}+s_{j(j+1)}+s_{j(j+2)}+\cdots+s_{j(n-2)}) \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,i_{j+1},\ldots,i_{n-2},i_j,n) \\
&+ (s_{j(n-1)}+s_{j(j+1)}+s_{j(j+2)}+\cdots+s_{j(n-3)}) \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,i_{j+1},\ldots,i_{n-3},i_j,i_{n-2},n) + \cdots \\
&+ s_{j(n-1)} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,i_j,i_{j+1},\ldots,i_{n-2},n) \Big]
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
&= \sum_{\alpha'} \mathcal{S}[\alpha'_{i_2,i_{j-1}}|i_2,\ldots,i_{j-1}]_{p_1} \\
&\times \left[\frac{\widetilde{\mathcal{S}}[i_j,i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-2},i_j]_{p_{n-1}}}{\widetilde{\mathcal{S}}[i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-2}]_{p_{n-1}}} \widetilde{A}_n(\alpha'_{i_2,\ldots,i_{j-1}},1,n-1,i_{j+1},\ldots,i_{n-2},i_j,n) \right. \\
&+ \frac{\widetilde{\mathcal{S}}[i_j,i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-3},i_j,i_{n-2}]_{p_{n-1}}}{\widetilde{\mathcal{S}}[i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-2}]_{p_{n-1}}} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,i_{j+1},\ldots,i_{n-3},i_j,i_{n-2},n) + \cdots \\
&+ \left. \frac{\widetilde{\mathcal{S}}[i_j,i_{j+1},\ldots,i_{n-2}|i_j,i_{j+1},\ldots,i_{n-2}]_{p_{n-1}}}{\widetilde{\mathcal{S}}[i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-2}]_{p_{n-1}}} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,i_j,i_{j+1},\ldots,i_{n-2},n) \right] .
\end{aligned}
\tag{B.3}
$$

Multiplying both sides with $\widetilde{\mathcal{S}}[i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-2}]_{p_{n-1}}$, we get the relation

$$
\begin{aligned}
&\sum_{\alpha} \mathcal{S}[\alpha_{i_2,i_j}|i_2,\ldots,i_j]_{p_1} \widetilde{\mathcal{S}}[i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-2}]_{p_{n-1}} \widetilde{A}_n(\alpha_{i_2,i_j},1,n-1,i_{j+1},\ldots,i_{n-2},n) \\
&= \sum_{\alpha'} \mathcal{S}[\alpha'_{i_2,i_{j-1}}|i_2,\ldots,i_{j-1}]_{p_1} \\
&\times \left[\widetilde{\mathcal{S}}[i_j,i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-2},i_j]_{p_{n-1}} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,i_{j+1},\ldots,i_{n-2},i_j,n) \right. \\
&+ \widetilde{\mathcal{S}}[i_j,i_{j+1},\ldots,i_{n-2}|i_{j+1},\ldots,i_{n-3},i_j,i_{n-2}]_{p_{n-1}} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,i_{j+1},\ldots,i_{n-3},i_j,i_{n-2},n) + \cdots \\
&+ \left. \widetilde{\mathcal{S}}[i_j,i_{j+1},\ldots,i_{n-2}|i_j,i_{j+1},\ldots,i_{n-2}]_{p_{n-1}} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,i_j,i_{j+1},\ldots,i_{n-2},n) \right] .
\end{aligned}
\tag{B.4}
$$

We then add all permutations of the legs $j+1,\ldots,n-2$. On the left-hand side we get

$$
\sum_{\alpha,\beta} \mathcal{S}[\alpha_{i_2,i_j}|i_2,\ldots,i_j]_{p_1} \widetilde{\mathcal{S}}[i_{j+1},\ldots,i_{n-2}|\beta_{i_{j+1},i_{n-2}}]_{p_{n-1}} \widetilde{A}_n(\alpha_{i_2,i_j},1,n-1,\beta_{i_{j+1},i_{n-2}},n) \,,
\tag{B.5}
$$

and on the right-hand side, since we have all permutations of i_j with the $\{j+1,\ldots,n-2\}$ set, we can write it as all permutations of $\{i_j,i_{j+1},\ldots,i_{n-2}\}$

$$
\sum_{\alpha',\beta'} \mathcal{S}[\alpha'_{i_2,i_{j-1}}|i_2,\ldots,i_{j-1}]_{p_1} \widetilde{\mathcal{S}}[i_j,i_{j+1},\ldots,i_{n-2}|\beta'_{i_j,i_{n-2}}]_{p_{n-1}} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,\beta'_{i_j,i_{n-2}},n) \,,
\tag{B.6}
$$

and we finally obtain the relation

$$
\begin{aligned}
&\sum_{\alpha,\beta} \mathcal{S}[\alpha_{i_2,i_j}|i_2,\ldots,i_j]_{p_1} \widetilde{\mathcal{S}}[i_{j+1},\ldots,i_{n-2}|\beta_{i_{j+1},i_{n-2}}]_{p_{n-1}} \widetilde{A}_n(\alpha_{i_2,i_j},1,n-1,\beta_{i_{j+1},i_{n-2}},n) \\
&= \sum_{\alpha',\beta'} \mathcal{S}[\alpha'_{i_2,i_{j-1}}|i_2,\ldots,i_{j-1}]_{p_1} \widetilde{\mathcal{S}}[i_j,i_{j+1},\ldots,i_{n-2}|\beta'_{i_j,i_{n-2}}]_{p_{n-1}} \widetilde{A}_n(\alpha'_{i_2,i_{j-1}},1,n-1,\beta'_{i_j,i_{n-2}},n) \,.
\end{aligned}
\tag{B.7}
$$

Bibliography

[1] Ronald Kleiss and Hans Kuijf. Multi - gluon cross-sections and five jet production at hadron colliders. *Nucl.Phys.*, B312:616, 1989.

[2] Michelangelo L. Mangano and Stephen J. Parke. Multiparton amplitudes in gauge theories. *Phys.Rept.*, 200:301–367, 1991.

[3] Stephen J. Parke and T.R. Taylor. An Amplitude for n Gluon Scattering. *Phys.Rev.Lett.*, 56:2459, 1986.

[4] Edward Witten. Perturbative gauge theory as a string theory in twistor space. *Commun.Math.Phys.*, 252:189–258, 2004.

[5] Freddy Cachazo, Peter Svrcek, and Edward Witten. MHV vertices and tree amplitudes in gauge theory. *JHEP*, 0409:006, 2004.

[6] Kasper Risager. A Direct proof of the CSW rules. *JHEP*, 0512:003, 2005.

[7] Ruth Britto, Freddy Cachazo, and Bo Feng. New recursion relations for tree amplitudes of gluons. *Nucl.Phys.*, B715:499–522, 2005.

[8] Ruth Britto, Freddy Cachazo, Bo Feng, and Edward Witten. Direct proof of tree-level recursion relation in Yang-Mills theory. *Phys.Rev.Lett.*, 94:181602, 2005.

[9] James Bedford, Andreas Brandhuber, Bill J. Spence, and Gabriele Travaglini. A Recursion relation for gravity amplitudes. *Nucl.Phys.*, B721:98–110, 2005.

[10] Nima Arkani-Hamed, Freddy Cachazo, and Jared Kaplan. What is the Simplest Quantum Field Theory? *JHEP*, 1009:016, 2010.

[11] Andreas Brandhuber, Paul Heslop, and Gabriele Travaglini. A Note on dual superconformal symmetry of the N=4 super Yang-Mills S-matrix. *Phys.Rev.*, D78:125005, 2008.

[12] Z. Bern, J.J.M. Carrasco, and Henrik Johansson. New Relations for Gauge-Theory Amplitudes. *Phys.Rev.*, D78:085011, 2008.

[13] Vittorio Del Duca, Lance J. Dixon, and Fabio Maltoni. New color decompositions for gauge amplitudes at tree and loop level. *Nucl.Phys.*, B571:51–70, 2000.

[14] Thomas Sondergaard. New Relations for Gauge-Theory Amplitudes with Matter. *Nucl.Phys.*, B821:417–430, 2009.

[15] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, and Pierre Vanhove. Minimal Basis for Gauge Theory Amplitudes. *Phys.Rev.Lett.*, 103:161602, 2009.

[16] S. Stieberger. Closed vs. Pure Open String Disk Amplitudes. 2009.

[17] Bo Feng, Rijun Huang, and Yin Jia. Gauge Amplitude Identities by On-shell Recursion Relation in S-matrix Program. *Phys.Lett.*, B695:350–353, 2011.

[18] Yin Jia, Rijun Huang, and Chang-Yong Liu. $U(1)$-decoupling, KK and BCJ relations in $\mathcal{N}=4$ SYM. *Phys.Rev.*, D82:065001, 2010.

[19] H. Kawai, D.C. Lewellen, and S.H.H. Tye. A Relation Between Tree Amplitudes of Closed and Open Strings. *Nucl.Phys.*, B269:1, 1986.

[20] Michael B. Green, John H. Schwarz, and Lars Brink. N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories. *Nucl.Phys.*, B198:474–492, 1982.

[21] Frits A. Berends, W.T. Giele, and H. Kuijf. On relations between multi - gluon and multigraviton scattering. *Phys.Lett.*, B211:91, 1988.

[22] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, Bo Feng, and Thomas Sondergaard. Gravity and Yang-Mills Amplitude Relations. *Phys.Rev.*, D82:107702, 2010.

[23] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, Bo Feng, and Thomas Sondergaard. Proof of Gravity and Yang-Mills Amplitude Relations. *JHEP*, 1009:067, 2010.

[24] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, Bo Feng, and Thomas Sondergaard. New Identities among Gauge Theory Amplitudes. *Phys.Lett.*, B691:268–273, 2010.

[25] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, Bo Feng, and Thomas Sondergaard. Unusual identities for QCD at tree-level. *J.Phys.Conf.Ser.*, 287:012030, 2011.

[26] Bo Feng and Song He. KLT and New Relations for N=8 SUGRA and N=4 SYM. *JHEP*, 1009:043, 2010.

[27] Henriette Elvang and Michael Kiermaier. Stringy KLT relations, global symmetries, and $E_{7(7)}$ violation. *JHEP*, 1010:108, 2010.

[28] Poul H. Damgaard, Rijun Huang, Thomas Sondergaard, and Yang Zhang. The Complete KLT-Map Between Gravity and Gauge Theories. 2012.

[29] Lance J. Dixon. Calculating scattering amplitudes efficiently. 1996.

[30] Michael E. Peskin. Simplifying Multi-Jet QCD Computation. 2011.

[31] Bryce S. DeWitt. Quantum Theory of Gravity. 3. Applications of the Covariant Theory. *Phys.Rev.*, 162:1239–1256, 1967.

[32] Sigurd Sannan. Gravity as the limit of the type II superstring theory. *Phys.Rev.*, D34:1749, 1986.

[33] Marc H. Goroff and Augusto Sagnotti. The Ultraviolet Behavior of Einstein Gravity. *Nucl.Phys.*, B266:709, 1986.

[34] A.E.M. van de Ven. Two loop quantum gravity. *Nucl.Phys.*, B378:309–366, 1992.

[35] Gerard 't Hooft and M.J.G. Veltman. One loop divergencies in the theory of gravitation. *Annales Poincare Phys.Theor.*, A20:69–94, 1974.

[36] Paolo Benincasa and Freddy Cachazo. Consistency Conditions on the S-Matrix of Massless Particles. 2007.

[37] Marcus T. Grisaru, H.N. Pendleton, and P. van Nieuwenhuizen. Supergravity and the S Matrix. *Phys.Rev.*, D15:996, 1977.

[38] Marcus T. Grisaru and H.N. Pendleton. Some Properties of Scattering Amplitudes in Supersymmetric Theories. *Nucl.Phys.*, B124:81, 1977.

[39] Dung Nguyen, Marcus Spradlin, Anastasia Volovich, and Congkao Wen. The Tree Formula for MHV Graviton Amplitudes. *JHEP*, 1007:045, 2010.

[40] Frits A. Berends and W.T. Giele. Recursive Calculations for Processes with n Gluons. *Nucl.Phys.*, B306:759, 1988.

[41] Zvi Bern, N.E.J. Bjerrum-Bohr, and David C. Dunbar. Inherited twistor-space structure of gravity loop amplitudes. *JHEP*, 0505:056, 2005.

[42] Freddy Cachazo and Peter Svrcek. Tree level recursion relations in general relativity. 2005.

[43] N.E.J. Bjerrum-Bohr, David C. Dunbar, Harald Ita, Warren B. Perkins, and Kasper Risager. MHV-vertices for gravity amplitudes. *JHEP*, 0601:009, 2006.

[44] N.E.J. Bjerrum-Bohr, David C. Dunbar, Harald Ita, Warren B. Perkins, and Kasper Risager. The No-Triangle Hypothesis for N=8 Supergravity. *JHEP*, 0612:072, 2006.

[45] Paolo Benincasa, Camille Boucher-Veronneau, and Freddy Cachazo. Taming Tree Amplitudes In General Relativity. *JHEP*, 0711:057, 2007.

[46] Z. Bern, J.J. Carrasco, D. Forde, H. Ita, and Henrik Johansson. Unexpected Cancellations in Gravity Theories. *Phys.Rev.*, D77:025010, 2008.

[47] Nima Arkani-Hamed and Jared Kaplan. On Tree Amplitudes in Gauge Theory and Gravity. *JHEP*, 0804:076, 2008.

[48] N.E.J. Bjerrum-Bohr and Pierre Vanhove. Absence of Triangles in Maximal Supergravity Amplitudes. *JHEP*, 0810:006, 2008.

[49] Simon Badger, N.E.J. Bjerrum-Bohr, and Pierre Vanhove. Simplicity in the Structure of QED and Gravity Amplitudes. *JHEP*, 0902:038, 2009.

[50] Fernando Quevedo, Sven Krippendorf, and Oliver Schlotterer. Cambridge Lectures on Supersymmetry and Extra Dimensions. 2010.

[51] Massimo Bianchi, Henriette Elvang, and Daniel Z. Freedman. Generating Tree Amplitudes in N=4 SYM and N = 8 SG. *JHEP*, 0809:063, 2008.

[52] Henriette Elvang, Yu-tin Huang, Cheng Peng, Yu-tin Huang, and Cheng Peng. On-shell superamplitudes in N<4 SYM. *JHEP*, 1109:031, 2011.

[53] Z. Bern, J.J.M. Carrasco, H. Ita, H. Johansson, and R. Roiban. On the Structure of Supersymmetric Sums in Multi-Loop Unitarity Cuts. *Phys.Rev.*, D80:065029, 2009.

[54] M.B. Green, J.H. Schwarz, and E. Witten. Superstring Theory. *Cambridge, UK, University Press*, 1987.

[55] Bo Feng and Mingxing Luo. An Introduction to On-shell Recursion Relations. 2011.

[56] Yi-Xin Chen, Yi-Jian Du, and Bo Feng. A Proof of the Explicit Minimal-basis Expansion of Tree Amplitudes in Gauge Field Theory. *JHEP*, 1102:112, 2011.

[57] S.H. Henry Tye and Yang Zhang. Dual Identities inside the Gluon and the Graviton Scattering Amplitudes. *JHEP*, 1006:071, 2010.

[58] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, Thomas Sondergaard, and Pierre Vanhove. Monodromy and Jacobi-like Relations for Color-Ordered Amplitudes. *JHEP*, 1006:003, 2010.

[59] Carlos R. Mafra. Simplifying the Tree-level Superstring Massless Five-point Amplitude. *JHEP*, 1001:007, 2010.

[60] Carlos R. Mafra, Oliver Schlotterer, and Stephan Stieberger. Explicit BCJ Numerators from Pure Spinors. *JHEP*, 1107:092, 2011.

[61] Zvi Bern and Tristan Dennen. A Color Dual Form for Gauge-Theory Amplitudes. *Phys.Rev.Lett.*, 107:081601, 2011.

[62] Ricardo Monteiro and Donal O'Connell. The Kinematic Algebra From the Self-Dual Sector. *JHEP*, 1107:007, 2011.

[63] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, Ricardo Monteiro, and Donal O'Connell. Algebras for Amplitudes. *JHEP*, 1206:061, 2012.

[64] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, Thomas Sondergaard, and Pierre Vanhove. The Momentum Kernel of Gauge and Gravity Theories. *JHEP*, 1101:001, 2011.

[65] Thomas Sondergaard. Perturbative Gravity and Gauge Theory Relations: A Review. *Adv.High Energy Phys.*, 2012:726030, 2012.

[66] Z. Bern, A. De Freitas, and H.L. Wong. On the coupling of gravitons to matter. *Phys.Rev.Lett.*, 84:3531, 2000.

[67] Yi-Jian Du, Bo Feng, and Chih-Hao Fu. BCJ Relation of Color Scalar Theory and KLT Relation of Gauge Theory. *JHEP*, 1108:129, 2011.

[68] Z. Bern, Lance J. Dixon, M. Perelstein, and J.S. Rozowsky. Multileg one loop gravity amplitudes from gauge theory. *Nucl.Phys.*, B546:423–479, 1999.

[69] Steven Weinberg. Infrared photons and gravitons. *Phys.Rev.*, 140:B516–B524, 1965.

[70] Frits A. Berends and W.T. Giele. Multiple Soft Gluon Radiation in Parton Processes. *Nucl.Phys.*, B313:595, 1989.

[71] Bo Feng, Song He, Rijun Huang, and Yin Jia. Note on New KLT relations. *JHEP*, 1010:109, 2010.

[72] Thomas Sondergaard. Gravity and Yang-Mills amplitude relations in field theory. *Nucl.Phys.Proc.Suppl.*, 216:268–269, 2011.

[73] Henry Tye and Yang Zhang. Comment on the Identities of the Gluon Tree Amplitudes. 2010.

[74] Zvi Bern, Tristan Dennen, Yu-tin Huang, and Michael Kiermaier. Gravity as the Square of Gauge Theory. *Phys.Rev.*, D82:065003, 2010.

[75] E. Cremmer and B. Julia. The SO(8) Supergravity. *Nucl.Phys.*, B159:141, 1979.

[76] Bo Feng, Yin Jia, and Rijun Huang. Relations of loop partial amplitudes in gauge theory by Unitarity cut method. *Nucl.Phys.*, B854:243–275, 2012.

[77] Stephen G. Naculich. All-loop group-theory constraints for color-ordered SU(N) gauge-theory amplitudes. *Phys.Lett.*, B707:191–197, 2012.

[78] Alexander C. Edison and Stephen G. Naculich. SU(N) group-theory constraints on color-ordered five-point amplitudes at all loop orders. *Nucl.Phys.*, B858:488–501, 2012.

[79] Alexander C. Edison and Stephen G. Naculich. Symmetric-group decomposition of SU(N) group-theory constraints on four-, five-, and six-point color-ordered amplitudes. 2012.

[80] Rutger H. Boels and Reinke Sven Isermann. New relations for scattering amplitudes in Yang-Mills theory at loop level. *Phys.Rev.*, D85:021701, 2012.

[81] Rutger H. Boels and Reinke Sven Isermann. Yang-Mills amplitude relations at loop level from non-adjacent BCFW shifts. *JHEP*, 1203:051, 2012.

[82] Yi-Jian Du and Hui Luo. On General BCJ Relation at One-loop Level in Yang-Mills Theory. 2012.

[83] Zvi Bern, Gordon Chalmers, Lance J. Dixon, and David A. Kosower. One loop N gluon amplitudes with maximal helicity violation via collinear limits. *Phys.Rev.Lett.*, 72:2134–2137, 1994.

[84] N.E.J. Bjerrum-Bohr, P.H. Damgaard, H. Johansson, and T. Sondergaard. Monodromy–like Relations for Finite Loop Amplitudes. *JHEP*, 1105:039, 2011.

[85] Zvi Bern and David A. Kosower. The Computation of loop amplitudes in gauge theories. *Nucl.Phys.*, B379:451–561, 1992.

[86] Zvi Bern, Lance J. Dixon, and David A. Kosower. New QCD results from string theory. 1993.

[87] Zvi Bern, Lance J. Dixon, and David A. Kosower. One loop corrections to five gluon amplitudes. *Phys.Rev.Lett.*, 70:2677–2680, 1993.

[88] Gregory Mahlon. Multi - gluon helicity amplitudes involving a quark loop. *Phys.Rev.*, D49:4438–4453, 1994.

[89] Zvi Bern, Lance J. Dixon, and David A. Kosower. The last of the finite loop amplitudes in QCD. *Phys.Rev.*, D72:125003, 2005.

[90] Zvi Bern, John Joseph M. Carrasco, and Henrik Johansson. Perturbative Quantum Gravity as a Double Copy of Gauge Theory. *Phys.Rev.Lett.*, 105:061602, 2010.

[91] Z. Bern, J.J.M. Carrasco, and H. Johansson. The Structure of Multiloop Amplitudes in Gauge and Gravity Theories. *Nucl.Phys.Proc.Suppl.*, 205-206:54–60, 2010.

[92] Z. Bern, J.J.M. Carrasco, Lance J. Dixon, H. Johansson, and R. Roiban. The Complete Four-Loop Four-Point Amplitude in N=4 Super-Yang-Mills Theory. *Phys.Rev.*, D82:125040, 2010.

[93] Zvi Bern, John Joseph Carrasco, Lance J. Dixon, Henrik Johansson, and Radu Roiban. Amplitudes and Ultraviolet Behavior of N = 8 Supergravity. 2011.

[94] John Joseph M. Carrasco and Henrik Johansson. Generic multiloop methods and application to N=4 super-Yang-Mills. *J.Phys.A*, A44:454004, 2011. 42 pages, 18 figures, invited review for a special issue of Journal of Physics A devoted to.

[95] John JosephM. Carrasco and Henrik Johansson. Five-Point Amplitudes in N=4 Super-Yang-Mills Theory and N=8 Supergravity. *Phys.Rev.*, D85:025006, 2012.

[96] Z. Bern, C. Boucher-Veronneau, and H. Johansson. N >= 4 Supergravity Amplitudes from Gauge Theory at One Loop. *Phys.Rev.*, D84:105035, 2011.

[97] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson, and R. Roiban. Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes. *Phys.Rev.*, D85:105014, 2012.

[98] Zvi Bern, Lance J. Dixon, David C. Dunbar, and David A. Kosower. One loop n point gauge theory amplitudes, unitarity and collinear limits. *Nucl.Phys.*, B425:217–260, 1994.

[99] Z. Bern, J.J. Carrasco, Lance J. Dixon, H. Johansson, and R. Roiban. The Ultraviolet Behavior of N=8 Supergravity at Four Loops. *Phys.Rev.Lett.*, 103:081301, 2009.

[100] Yi-Xin Chen, Yi-Jian Du, and Qian Ma. Relations Between Closed String Amplitudes at Higher-order Tree Level and Open String Amplitudes. *Nucl.Phys.*, B824:314–330, 2010.

[101] Yi-Xin Chen, Yi-Jian Du, and Qian Ma. Disk relations for tree amplitudes in minimal coupling theory of gauge field and gravity. *Nucl.Phys.*, B833:28–45, 2010.

[102] Yi-Xin Chen, Yi-Jian Du, and Bo Feng. On tree amplitudes with gluons coupled to gravitons. *JHEP*, 1101:081, 2011.

[103] Pawel Caputa and Shinji Hirano. Observations on Open and Closed String Scattering Amplitudes at High Energies. *JHEP*, 1202:111, 2012.

[104] N.E.J. Bjerrum-Bohr and Oluf Tang Engelund. Gravitino Interactions from Yang-Mills Theory. *Phys.Rev.*, D81:105009, 2010.

[105] Andrew Hodges. A simple formula for gravitational MHV amplitudes. 2012.

[106] Bo Feng and Song He. Graphs, determinants and gravity amplitudes. 2012.

[107] Freddy Cachazo and Yvonne Geyer. A 'Twistor String' Inspired Formula For Tree-Level Scattering Amplitudes in N=8 SUGRA. 2012.

[108] Radu Roiban, Marcus Spradlin, and Anastasia Volovich. On the tree level S matrix of Yang-Mills theory. *Phys.Rev.*, D70:026009, 2004.

[109] Freddy Cachazo. Fundamental BCJ Relation in N=4 SYM From The Connected Formulation. 2012.